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were balloted for and duly elected Fellows of the Society.

On the Solution of the Equations in the Method of Least Squares. By J. W. L. Glaisher.

§ 1. In No. 404 (July 2, 1840) of the *Astronomische Nachrichten*, Bessel published some very elegant formulæ for the solution of the final equations in the method of least squares and the determination of the weights, when the number of unknowns amounted only to three, that had been communicated to him by Jacobi; and in a recent number of the same journal (No. 1960, October 8, 1873) Dr. H. Seeliger of the Leipzig Observatory, has inserted a proof of them. As, however, Dr. Seeliger has confined himself to merely verifying the formulæ, without referring to the points on which their special excellence and characteristic properties depend, I have thought it would be worth while to examine them from this point of view, omitting the proof, which presents no difficulty. And as being very closely connected with this investigation, I have added several

notes—forming the larger portion of the paper—which were suggested by Gauss's original memoirs referred to more particularly further on, and Encke's excellent *résumé* of the whole subject in the *Berliner Astronomische Jahrbücher* for 1834, 1835, and 1836.

§ 2. Writing the equations—

$$\begin{aligned}(aa)\,x + (ab)\,y + (ac)\,z &= (an)\\ (ab)\,x + (bb)\,y + (bc)\,z &= (bn)\\ (ac)\,x + (bc)\,y + (cc)\,z &= (cn)\end{aligned}$$

Jacobi's formulæ as given in No. 404 of the *Astr. Nach.* are—

$$r = \pm\sqrt{\pm\,(bc)\,(ac)\,(ab)}$$

$$\alpha = \frac{r}{(bc)}, \qquad \beta = \frac{r}{(ac)}, \qquad \gamma = \frac{r}{(ab)}$$

$$A = (aa) \mp \alpha\alpha, \qquad B = (bb) \mp \beta\beta, \qquad C = (cc) \mp \gamma\gamma$$

$$\alpha' = \frac{\alpha}{A}, \qquad \beta' = \frac{\beta}{B}, \qquad \gamma' = \frac{\gamma}{C}$$

$$R = 1 \pm (\alpha\alpha' + \beta\beta' + \gamma\gamma')$$

$$a = \frac{(an)}{R}, \qquad b = \frac{(bn)}{R}, \qquad c = \frac{(cn)}{R}$$

$$\rho = \alpha' a + \beta' b + \gamma' c$$

$$x = \frac{(an)}{A} \mp \alpha'\rho, \qquad y = \frac{(bn)}{B} \mp \beta'\rho, \qquad z = \frac{(cn)}{C} \mp \gamma'\rho,$$

$$X = \frac{RA}{R \mp \alpha\alpha'}, \qquad Y = \frac{RB}{R \mp \beta\beta'}, \qquad Z = \frac{RC}{R \mp \gamma\gamma'}$$

$$(vv) = (nn) - \frac{(an)^2}{A} - \frac{(bn)^2}{B} - \frac{(cn)^2}{C} \pm R\rho\rho\,*$$

X, Y, Z being the weights, (vv) the sum of the squares of the residuals and the upper or lower signs being taken throughout according to the sign of the product under the radical in r.† Bessel adds, that Jacobi's formulæ give x, y, z and X, Y, Z with only 32 logarithmic entries, while Gauss's method requires 23 entries in order to obtain x, y, z, and 12 more to obtain X, Y, Z: a remark that will be noticed further on.

§ 3. In discussing the formulæ, when the number of un-

* Bessel, by a slip, writes :—

$$(vv) = (nn) - \frac{(an)^2}{A} - \frac{(bn)^2}{B} - \frac{(cn)^2}{C} \pm \frac{\rho\rho}{R}$$

(*Astr. Nach.*, *No.* 404).

† It may save the reader of Bessel's paper (*Astr. Nach.*, No. 404) a few moments' uncertainty to remark that in the statement of the data of the example there worked out, two of the coefficients are accidentally interchanged—viz., they should be, $(ac) = 28{\cdot}4660$ and $(bc) = 3{\cdot}3000$.

knowns is only three, it is more convenient to write the original equations in their usual mathematical form, viz.:

$$\begin{aligned} ax + hy + gz &= \alpha \\ hx + by + fz &= \beta \\ gx + fy + cz &= \gamma \end{aligned}$$

and to take

$$\nabla = \begin{vmatrix} a, & h, & g, \\ h, & b, & f, \\ g, & f, & c, \end{vmatrix}.$$

Further, let great letters denote the minors of the corresponding small letters, so that

$$\begin{aligned} &A = bc - f^2, \quad B = ca - g^2, \quad C = ab - h^2 \\ &F = gh - af, \quad G = hf - bg, \quad H = fg - ch \end{aligned}$$

then

$$\begin{aligned} &\nabla a = BC - F^2, \quad \nabla b = \&c., \&c.; \quad \nabla f = GH - AF, \quad \nabla g = \&c., \&c. \\ &\nabla x = A\alpha + H\beta + G\gamma, \quad \nabla y = \&c., \&c. \\ &\nabla = A a + H h + G g = \&c. \end{aligned}$$

and, for the weights,

$$X = \frac{\nabla}{A}, \qquad Y = \frac{\nabla}{B}, \qquad Z = \frac{\nabla}{C}.$$

It will then be readily seen that the following is a free translation of Jacobi's process, in which the numbers of logarithms and anti-logarithms required are left unaltered; although, for the sake of clearness, certain factors are attached to different quantities from those to which they belong in Jacobi's formulæ, and several signs are changed.

Form the quantities:

$$\log f, \qquad \log g, \qquad \log h$$

$$\frac{F}{f} = \frac{gh}{f} - a, \qquad \frac{G}{g} = \frac{hf}{g} - b, \qquad \frac{H}{h} = \frac{fg}{h} - c,$$

$$\log\left(\frac{F}{f}\right), \qquad \log\left(\frac{G}{g}\right), \qquad \log\left(\frac{H}{h}\right)$$

$$\log F, \qquad \log G, \qquad \log H$$

$$\nabla' = \frac{gh}{F} + \frac{hf}{G} + \frac{fg}{H} - 1, \quad \log \nabla'$$

$$\log \alpha, \qquad \log \beta, \qquad \log \gamma$$

$$\rho = \frac{\alpha}{\mathrm{F}} + \frac{\beta}{\mathrm{G}} + \frac{\gamma}{\mathrm{H}}, \quad \log \rho$$

$$x = -\frac{\alpha f}{\mathrm{F}} + \frac{f g h}{\nabla' \mathrm{F}}\rho, \quad y = -\frac{\beta g}{\mathrm{G}} + \frac{f g h}{\nabla' \mathrm{G}}\rho, \quad z = -\frac{\gamma h}{\mathrm{H}} + \frac{f g h}{\nabla' \mathrm{H}}\rho$$

$$\log\left(\frac{g h}{\mathrm{F}} - \nabla'\right), \qquad \log\left(\frac{h f}{\mathrm{G}} - \nabla'\right), \qquad \log\left(\frac{f g}{\mathrm{H}} - \nabla'\right)$$

$$\mathrm{X} = \frac{\nabla' \mathrm{F}}{f\left(\frac{g h}{\mathrm{F}} - \nabla'\right)}, \qquad \mathrm{Y} = \frac{\nabla' \mathrm{G}}{g\left(\frac{h f}{\mathrm{G}} - \nabla'\right)}, \qquad \mathrm{Z} = \frac{\nabla' \mathrm{H}}{h\left(\frac{f g}{\mathrm{H}} - \nabla'\right)}$$

$$(v\,v) = (n\,n) + \frac{\alpha^2 f}{\mathrm{F}} + \frac{\beta^2 g}{\mathrm{G}} + \frac{\gamma^2 h}{\mathrm{H}} - \frac{f g h}{\nabla'}\rho^2.$$

In order to appreciate what Jacobi effects, it is to be observed that his object clearly was to diminish as much as possible the number of logarithmic entries requisite, at the same time keeping the formulæ symmetrical. Thus he endeavours to replace multiplications and divisions by additions and subtractions, as the latter do not entail reference to a table. If the equations were solved in a straightforward way by calculating A, B, C, F, G, H, and their logarithms, and substituting in the final equations $\nabla x = \mathrm{A}\alpha + \mathrm{H}\beta + \mathrm{G}\gamma$, &c., $\mathrm{X} = \frac{\nabla}{\mathrm{A}}$, &c., no less than forty-two entries (viz. sixteen logarithms and twenty-six anti-logarithms) would be required for the determination of x, y, z, X, Y, Z. But it is clear that one entry can be dispensed with in the calculation of such a pair of terms as $ab - cd$, by writing them in the form $b\,(a - \frac{cd}{b})$, so that the logarithm of a is not required. Thus, by calculating $\log \frac{\mathrm{F}}{f}$, &c., and thence log F, &c., in place of log F, &c., directly, Jacobi avoids the necessity of having to take out the logarithms of a, b, c. The quantity ∇' is connected with ∇ by the relation—

$$\nabla' = \frac{f g h}{\mathrm{F G H}}\nabla;$$

it can readily be shown that

$$\nabla = \frac{\mathrm{G H}}{f} + \frac{\mathrm{H F}}{g} + \frac{\mathrm{F G}}{h} - \frac{\mathrm{F G H}}{f g h};$$

and the factor $\frac{f g h}{\mathrm{F G H}}$ is introduced to replace the calculation of the last term, which would require the taking out of an anti-

logarithm, by the simple subtraction of unity. It will be observed that the calculation of $\frac{gh}{F} - \nabla'$, &c., requires no tabular entries, as the values of $\frac{gh}{F}$, $\frac{hf}{G}$ and $\frac{fg}{H}$ were obtained as steps in the calculation of ∇'.

The denominators of X, Y, Z must involve A, B, C as factors, and it is easily shown that

$$\frac{gh}{F} - \nabla' = \frac{gh}{GH}A, \quad \frac{hf}{G} - \nabla' = \frac{hf}{HF}B, \quad \frac{fg}{H} - \nabla' = \frac{fg}{FG}C.$$

As A, B, C, and their logarithms thus have ultimately to be calculated for the determination of the weights, it is natural to enquire whether it would not be more convenient to calculate them at an earlier stage of the process, and use their values in determination of x, y, z. In fact, after finding $\log \nabla'$ it seems as if it would be an improvement for the method to proceed—

$$\log \mu = \log\left(\frac{F}{f}\right) + \log\left(\frac{G}{g}\right) + \log\left(\frac{H}{h}\right), \log \nabla = \log \nabla' + \log \mu$$

$$A' = \frac{gh}{F} - \nabla', \quad B' = \frac{hf}{G} - \nabla', \quad C' = \frac{fg}{H} - \nabla'$$

$$\log A = \log \mu + \log f + \log A' - \log F, \log B = \&c., \log C = \&c.$$

$$\log \alpha, \quad \log \beta, \quad \log \gamma$$

$$x = \frac{A\alpha}{\nabla} + \frac{H\beta}{\nabla} + \frac{G\gamma}{\nabla}, \; X = \frac{\nabla}{A}$$

$$y = \&c., z = \&c., Y = \&c., Z = \&c.$$

By this means we should also save one entry, as with Jacobi $\log \rho$ requires 4 entries, and x, y, z, 6 more, while in the scheme just written, x, y, z, merely require 9 anti-logarithms. Thus, x, y, z, X, Y, Z, are obtained by 31 entries, one less than Jacobi needs; but from one point of view Bessel's comparison is scarcely fair to Jacobi, unless the calculation of $(v\,v)$ is also included. There can be no doubt that the quantity ρ was introduced for the sake of saving entries in the formation of $(v\,v)$, for although one entry more is required in finding x, y, z, two are avoided in calculating $(v\,v)$, which according, to the second scheme, would have to be obtained from—

$$(v\,v) = (n\,n) - \alpha x - \beta y - \gamma z$$

and as the logarithms of x, y, z, still have to be found, 6 entries are required, while Jacobi's method only needs 4. He therefore gains one on the whole process, which requires 36 entries, while the other scheme requires 37. If the values of x, y, z, had been

found by Gauss's method of substitution, the value of (vv) or $(nn.3)$ would need only 3 additional entries, so that no doubt Bessel made the comparison end where he did, as he thus broke off at the point most favourable to Jacobi. But unless the whole process is included, no sufficient reason appears why ρ should have been introduced. On the whole then, Jacobi requires 36 and Gauss 38 entries. In giving the number of additional entries required in finding X, Y, Z, as 12, Bessel was correct; as he refers to § 32 of the *Theoria Combinationis Observationum*, but had he supposed the weights determined in the manner pointed out by Encke, on p. 314 of the *Berl. Astr. Jahrbuch* for 1835 (which is the method reproduced by Chauvenet in his *Astronomy*, vol. ii. p. 537, fourth edition, 1868), only 7 additional entries would be required, and the total number would thus become 33, or 3 less than with Jacobi. The latter, however, has the advantage of symmetry.

§ 3. It is perhaps worth while to here add, for the sake of comparison, the solution of the equations by the method of substitution, with the determination of the weights by Encke's method, using, as far as possible, the same letters as those already employed.

$$\begin{aligned} ax + hy + gz &= \alpha \\ hx + by + fz &= \beta \\ gx + fy + cz &= \gamma \end{aligned}$$

therefore

$$x = \frac{\alpha}{a} - \frac{h}{a}y - \frac{g}{a}z \quad \ldots\ldots\ldots\ldots \quad (1)$$

and

$$y\left(b - \frac{h^2}{a}\right) + z\left(f - \frac{hg}{a}\right) = \beta - \frac{h\alpha}{a},$$

$$y\left(f - \frac{gh}{a}\right) + z\left(c - \frac{g^2}{a}\right) = \gamma - \frac{g\alpha}{a}$$

viz.

$$y\frac{\mathrm{C}}{a} - z\frac{\mathrm{F}}{a} = \beta'$$

$$-y\frac{\mathrm{F}}{a} + z\frac{\mathrm{B}}{a} = \gamma'$$

which write

$$\begin{aligned} y\,\mathrm{C}' - z\,\mathrm{F}' &= \beta', \\ -y\,\mathrm{F}' + z\,\mathrm{B}' &= \gamma' \end{aligned}$$

whence

$$y = \frac{\mathrm{F}'}{\mathrm{C}'}z + \frac{\beta'}{\mathrm{C}'}, \quad \ldots\ldots\ldots\ldots \quad (2)$$

and

$$z\left(\mathrm{B}' - \frac{\mathrm{F}'^2}{\mathrm{C}'}\right) = \gamma' + \frac{\mathrm{F}'\beta'}{\mathrm{C}'}, \quad \ldots\ldots\ldots \quad (3)$$

(3), (2) and (1) give x, y, z, with 23 entries.

$$\mathrm{Z} = \mathrm{B}' - \frac{\mathrm{F}'^2}{\mathrm{C}'} = \mathrm{C}'' \text{ say}$$

$$\mathrm{Y} = \mathrm{C}' \cdot \frac{\mathrm{C}''}{\mathrm{B}'}$$

$$\mathrm{X} = a \cdot \frac{\mathrm{C}'}{b} \cdot \frac{\mathrm{C}''}{c - \frac{f^2}{b}}$$

$$(v\,v) = (n\,n) - \frac{\alpha^2}{a} - \frac{\beta'^2}{\mathrm{C}'} - \frac{\gamma''^2}{\mathrm{C}''}, \text{ where } \gamma'' = \gamma' + \frac{\mathrm{F}'\beta'}{\mathrm{C}'}$$

and the process altogether requires 14 logarithms and 19 anti-logarithms.

In the case of two unknowns it can be shown that the method of substitution gives x, y, X, Y, by twelve entries and $(v\,v)$ by two more, while a strictly symmetrical procedure requires thirteen entries for x, y, X, Y, and four more for $(v\,v)$.

§ 4. The geometrical interpretation of the results of the method of least squares, when the number of unknowns amounts to three, possesses some interest.

Construct the quadric

$$(aa)x^2 + (bb)y^2 + (cc)z^2 + 2(bc)yz + 2(ca)zx + 2(ab)xy + 2(an)x + 2(bn)y + 2(cn)z + (nn) = 0 \quad \ldots \quad (p)$$

Then the most probable values of x, y, z are the coordinates of its centre; and if the quadric be transformed to the centre as origin, the axes remaining parallel, the equation (p) assumes the form

$$(aa)x^2 + (bb)y^2 + (cc)z^2 + 2(bc)yz + 2(ca)zx + 2(ab)xy + (vv) = 0 \quad \ldots \quad (q)$$

for, x', y', z', being the coordinates of the centre, $(v\,v) = \Sigma(ax' + by' + cz' + n)^2 = (a\,a)\,x'^2 +$ &c. The same thing is also seen to follow at once from the known result $(v\,v) = (a\,n)\,x' + (b\,n)\,y' + (c\,n)\,z' + (n\,n)$.

If in (q) we put $z = 0$, we find the area of the conic so obtained

$$= \pi \frac{(v\,v)}{\sqrt{\{(a\,a)(b\,b) - (a\,b)^2\}}} = \frac{\pi(v\,v)}{\sqrt{\nabla}}\sqrt{\mathrm{Z}}$$

so that the square roots of the weights of x, y, z are proportional to the areas of the central sections of (p), formed by planes drawn parallel to the coordinate axes; or what is the same thing, the

probable errors of x, y, z are inversely proportional to the areas of these sections.

Further, since the volume of the quadric $= \frac{4\pi}{3}\sqrt{\frac{(v\,v)^3}{\nabla}}$, it follows that everything else remaining constant except the values of the v's which are (or may be taken to be) the quantities directly observed, then $(v\,v)$, the sum of the squares of the residuals, varies as the superficial dimensions of the quadric; or, what is the same thing, the probable error of an observation varies as the linear dimensions of the quadric.

§ 5. In what follows, I shall adopt entirely the notation that Encke has used in the *Berliner Astronomisches Jahrbuch* for 1835, which is substantially the same as Gauss's; but I shall replace the square brackets, as in $[b\,b\,.\,1]$, by simple parentheses, as in $(b\,b\,.\,1)$, and omit even these in determinants or anywhere else where there is no risk of confusion. The only places that I know of in which Gauss has explained the method of solution of the equations that arise in the method of least squares are §§ 182–186 of the *Theoria Motus* (reprinted, *Werke*, vol. vii. pp. 237–241), the *Theoria Combinationis Observationum* and the *Supplementum* to it (*Werke*, vol. iv.), and the *Disquisitio de Elementis Ellipticis Palladis* (Comm. Soc. Gott., vol. i. 1808–1811), in which his characteristic notation and algorithm first appeared. This memoir has not yet, I believe, been reprinted in his collected works. Although, perhaps, there is not a great deal in Encke's essay that is not implicitly to be found in Gauss, still there is a considerable difference between his full and systematic exposition of the different methods, and the original memoirs in which some parts of the processes are rather sketched than worked out. It is, therefore, convenient to refer generally to the *Jahrbuch* rather than to Gauss's papers, on account of the uniform notation and treatment that characterises the former. The account given by Chauvenet, in his *Astronomy* (vol. ii. pp. 529–549, arts. 41–52, fourth edition), is almost wholly taken from Encke, the chief difference being that the number of unknowns is supposed to be four instead of six.

Writing the final* equations

$$\left.\begin{aligned}
(a\,a)\,x + (a\,b)\,y + (a\,c)\,z \ldots + (a\,n) &= 0\\
(b\,a)\,x + (b\,b)\,y + (b\,c)\,z \ldots + (b\,n) &= 0\\
(c\,a)\,x + (c\,b)\,y + (c\,c)\,z \ldots + (c\,n) &= 0\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \quad \ldots \text{(i)}$$

* These equations are sometimes called the *normal* equations, but the name does not appear very felicitous. The only objection to the word *final* is that Encke terms *Endgleichungen* the equations from which $x, y, z \ldots$ are immediately determined (*Jahrbuch*, 1835, p. 272).

multiplying them by Q, Q′, Q″ . . . and choosing Q, Q′, Q″, . . . so that

$$\begin{aligned}(a\,a)\,Q + (a\,b)\,Q' + (a\,c)\,Q'' + \ldots &= 1\\ (b\,a)\,Q + (b\,b)\,Q' + (b\,c)\,Q'' + \ldots &= 0 \qquad \ldots \text{ ii)}\\ (c\,a)\,Q + (c\,b)\,Q' + (c\,c)\,Q'' + \ldots &= 0\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

then

$$\begin{aligned}-x &= (a\,n)\,Q + (b\,n)\,Q' + (c\,n)\,Q'' + \ldots\\ &= (a_1\,Q + b_1\,Q' + c_1\,Q'' \ldots)\,n_1 + (a_2 Q + b_2 Q' + c_2 Q'' \ldots)\,n_2 + \ldots\\ &= a_1\,n_1 + a_2 n_2 + \ldots\end{aligned}$$

and X, the weight of x, $= \dfrac{1}{a_1^{\,2} + a_2^{\,2} + \ldots} = \dfrac{1}{(a\,a)}$

The value of $a\,a$ is usually obtained by means of an elegant, though indirect procedure, by which it is shown that $(a\,a) = Q$, so that the weight of x is found by replacing $(a\,n)$ by -1, and $(b\,n)$, $(c\,n)$, &c. by zeros, in (i); and then solving for x, the value of which is the reciprocal of X. This want of directness is a distinct disadvantage, as the natural course would be to find x, and therefrom calculate $(a\,a)$. On this point Encke writes: "Die unmittelbare Ableitung der linearen Form würde indessen, wenn jedes a gegeben werden sollte, der Verwickelung der Formeln wegen, theils höchst weitläufig, kaum ausführbar sein, theils auch unnütz, weil zu unserm Zwecke die Summe $[a\,a]$ allein erforderlich ist" (*Jahrbuch*, 1835, p. 289). It is true that a_1, a_2, &c. are not themselves wanted, but it is certainly of interest to exhibit x as a linear function of n_1, n_2, &c., and thence obtain the weights directly, so as to remove the somewhat artificial character that otherwise attaches to the investigation. This is easily effected by the use of determinants, for

$$x \begin{vmatrix} a\,a, & a\,b, & a\,c, & \ldots\\ b\,a, & b\,b, & b\,c, & \ldots\\ c\,a, & c\,b, & c\,c, & \ldots\\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = \begin{vmatrix} a\,n, & a\,b, & a\,c, & \ldots\\ b\,n, & b\,b, & b\,c, & \ldots\\ c\,n, & c\,b, & c\,c, & \ldots\\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$= n_1 \begin{vmatrix} a_1, & a\,b, & a\,c, & \ldots\\ b_1, & b\,b, & b\,c, & \ldots\\ c_1, & c\,b, & c\,c, & \ldots\\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} + n_2 \begin{vmatrix} a_2, & a\,b, & a\,c & \ldots\\ b_2, & b\,b, & b\,c & \ldots\\ c_2, & c\,b, & c\,c & \ldots\\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} + \&c.$$

since $a\,n$ or $(a\,n) = a_1\,n_1 + a_2\,n_2 + \ldots$

Denoting then, here and henceforth, the symmetrical determinant that multiplies x by ∇, and writing A, B, C, &c. for the minors of $a\,a$, $a\,b$, $a\,c$, . . . in ∇:

$$\nabla^2(aa) = (a_1\mathrm{A} + b_1\mathrm{B} + c_1\mathrm{C} + \ldots)^2 + (a_2\mathrm{A} + b_2\mathrm{B} + c_2\mathrm{C} + \ldots)^2 + \ldots$$
$$= \mathrm{A}\{\mathrm{A}(aa) + \mathrm{B}(ab) + \mathrm{C}(ac) + \ldots\}$$
$$+ \mathrm{B}\{\mathrm{A}(ab) + \mathrm{B}(bb) + \mathrm{C}(bc) + \ldots\} + \&c.$$

$$= \mathrm{A}\begin{vmatrix} aa, & ab, & ac \ldots \\ ba, & bb, & bc \ldots \\ ca, & cb, & cc \ldots \\ \ldots & \ldots & \ldots \ldots \end{vmatrix} + \mathrm{B}\begin{vmatrix} ab, & ab, & ac \ldots \\ bb, & bb, & bc \ldots \\ cb, & cb, & cc \ldots \\ \ldots & \ldots & \ldots \ldots \end{vmatrix} + \&c.$$

which expression reduces to its first term, as all the determinants that multiply B, C, D . . . have two columns identical; thus

$$(aa) = \frac{\mathrm{A}}{\nabla}$$

and we obtain (aa) in its simplest form, in which it is obviously equal to Q as determined from (ii).

It will have been noticed how well the notation $(a\,a)$, $(a\,b)$ . . . for the coefficients is adapted for use in the determinants; it being understood that the small letters $a, b, c, \ldots n$, without suffixes, have no meaning at all except in juxtaposition with one of themselves, we have practically Sylvester's umbral notation for the constituents of a determinant. Thus a and b are supposed unmeaning (except as types of $a_1, a_2 \ldots$ and $b_1, b_2 \ldots$), but $a\,b$ denotes $(a\,b)$ viz. $a_1 b_1 + a_2 b_2 + \ldots$

The subsidiary quantities $(b\,b\,.\,1)$, $(b\,c\,.\,1) \ldots (c\,c\,.\,2)$, $(c\,d\,.\,2) \ldots$ &c. (called by Encke *Hülfsgrössen* and by Chauvenet *Auxiliaries*), which occur in the process of solution of the equations by the method of substitution, admit of being expressed very simply as quotients of determinants, and it is interesting to examine what are their actual values in terms of the original coefficients. It is to be observed that the meaning of the notation is not definite until the order in which the letters $a, b, c \ldots$ are supposed to follow one another is fixed. Let, therefore, this order be that of the alphabet, so that, e.g.—

$$cd.1 = cd - \frac{ca.da}{aa},\quad cd.2 = cd.1 - \frac{(cb.1)(db.1)}{bb.1},\ \&c.$$

then

$$bb.1 = bb - \frac{ba.ba}{aa}$$

so that

$$(aa)(bb.1) = \begin{vmatrix} aa, & ab \\ ba, & bb \end{vmatrix}$$

Similarly

$$cc.2 = \frac{1}{bb.1}\begin{vmatrix} bb.1, & bc.1 \\ cb.1, & cc.1 \end{vmatrix}$$

$$= \frac{1}{bb.1} \begin{vmatrix} bb - \frac{ba.ba}{aa}, & bc - \frac{ba.ca}{aa} \\ cb - \frac{ca.ba}{aa}, & cc - \frac{ca.ca}{aa} \end{vmatrix}$$

$$= \frac{1}{(aa)(bb.1)} \begin{vmatrix} aa, & ab, & ac, \\ ba, & bb, & bc, \\ ca, & cb, & cc, \end{vmatrix}$$

and

$$(aa)(bb.1)(cc.2)(dd.3) = \begin{vmatrix} aa, & ab, & ac, & ad \\ ba, & bb, & bc, & bd \\ ca, & cb, & cc, & cd \\ da, & db, & dc, & dd \end{vmatrix}$$

The general law is now evident for the case when the pair of letters in the auxiliary are identical and correspond to the numeral postfixed: thus $ee.4$ is the quotient of two symmetrical determinants, the denominator being the determinant whose principal diagonal contains aa, bb, cc, dd, and the numerator being the same determinant bordered by an additional row and column, ea, eb, ec, ed, ee. Also, it can be shown in the same way that, for example—

$$(aa)(bb.1)(cc.2)(ef.3) = \begin{vmatrix} aa, & ab, & ac, & af \\ ba, & bb, & bc, & bf \\ ca, & cb, & cc, & cf \\ ea, & eb, & ec, & ef \end{vmatrix}$$

so that every auxiliary involving (say) the postfix 3 has the denominator $(aa)(bb.1)(cc.2)$, viz. the determinant whose principal diagonal is aa, bb, cc; while the numerator is this same determinant, bordered by the constituents formed by uniting the letters involved in the auxiliary with a, b, c, viz. in $ef.3$ the border is af, bf, cf, ef, ec, eb, ea.

The same rule holds good when the letter n is involved, and also when the auxiliary involves a pair of n's; thus

$$nn.2 = \begin{vmatrix} aa, & ab, & an \\ ba, & bb, & bn \\ na, & nb, & nn \end{vmatrix} \div \begin{vmatrix} aa, & ab \\ ba, & bb \end{vmatrix}$$

and the principle of the whole method of formation is clear. We see that the denominator is determined solely by the postfix, and that the pair of letters involved in the auxiliary determines the border, by the addition of which alone the numerator is distinguished from the denominator.

§ 7. In this section it is convenient to suppose with Encke that the number of unknowns amounts to six (x, y, z, w, u, t), although all the reasoning applies equally well whatever the number may be. Both Gauss (*Theoria Motus*, § 182 and *Disq. de Elem. Palladis*, p. 23) and Encke (*Jahrbuch*, p. 273) form the expression for the sum of the squares of the errors, viz.—

$$\Omega = (aa)x^2 \ldots + 2(ab)xy \ldots + 2(an)x \ldots + (nn)$$

and, with the view of making it a minimum, transform it into

$$\Omega = \frac{AA}{aa} + \frac{B'B'}{bb.1} + \frac{C''C''}{cc.2} + \frac{D'''D'''}{dd.3} + \frac{E^{iv}E^{iv}}{ee.4} + \frac{F^{v}F^{v}}{ff.5} + nn.6$$

where A is a linear function of all the unknowns, B′ is a linear function from which x is absent, C″ a linear function from which both x and y are absent, &c.

The transformation is practically the same as that employed by Mr. B. Williamson in the Appendix to his *Differential Calculus* (Dublin 1873), and in the *Quarterly Journal of Mathematics* (March 1872), for the determination of the minimum value of a quadratic expression, except that Gauss's and Encke's form is somewhat more general as the linear and constant terms are included. A, B′, C″ . . . are, in fact, *aux facteurs près*, the expressions which, when equated to zero, give the equations from which the unknowns are actually calculated, e.g.—

$$C'' = (cc.2)z + (cd.2)w + (ce.2)u + (cf.2)t + (cn.2)$$

that is to say—

$$\begin{vmatrix} aa, & ab \\ ba, & bb, \end{vmatrix} C'' = z \begin{vmatrix} aa, & ab, & ac, \\ ba, & bb, & bc, \\ ca, & cb, & cc, \end{vmatrix} + w \begin{vmatrix} aa, & ab, & ac, \\ ba, & bb, & bc \\ da, & db, & dc \end{vmatrix} + \&c.$$

Gauss's reasoning also shows that the minimum value of Ω is $nn.6$; and, from what has been said, it follows that

$$nn.6 = \frac{1}{\nabla} \begin{vmatrix} aa, & ab \ldots & af, & an \\ ba, & bb \ldots & bf, & bn \\ \ldots & \ldots & \ldots & \ldots \\ fa, & fb \ldots & ff, & fn \\ na, & nb \ldots & nf & nn \end{vmatrix} \quad \ldots\ldots\ldots \text{(iii)}$$

In his memoir on the orbit of *Pallas*, Gauss, for brevity, omitted the proof that aa, $(bb.1)$, $(cc.2)$ . . . are all positive, which is essential in order that the minimum value of Ω may be given by A = 0, B′ = 0, &c. On p. 277 of the *Jahrbuch*, Encke gives a proof; but perhaps the most direct method of demon-

strating this elegant result is by exhibiting $(a\,a)$, $(a\,a)\,(b\,b\,.\,1)$, &c. as sums of squares: thus $(a\,a)$ is positive, and so is $(b\,b\,.\,1)$, for

$$(a\,a)\,(b\,b.1) = \begin{vmatrix} aa, & ab \\ ba, & bb \end{vmatrix} = \Sigma \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}^2$$

Also

$$(a\,a)\,(b\,c.1) = \begin{vmatrix} aa, & ab \\ ca, & cb \end{vmatrix} = \Sigma \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} \begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix}$$

and

$$(b\,b.1)\,(c\,c.2) = \begin{vmatrix} bb.1, & bc.1 \\ bc.1, & cc.1 \end{vmatrix}$$

the constituents of this last determinant being of exactly the same form as in that to which $(a\,a)\,(b\,b\,.\,1)$ is equated; whence $(b\,b\,.\,1)\,(c\,c\,.\,2)$ is a sum of squares, and so on.

§ 8. The integral—

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots\ e^{-\Omega}\,dx\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv)}$$

evaluated by Gauss (*Theor. Mot.* § 182) and by Encke (*Jahrbuch*, p. 286) is when slightly transformed the same as that considered by Todhunter in his memoir "On the Method of Least Squares" (*Cambridge Phil. Trans.*, vol. xi.). The result obtained by Gauss and Encke for (iv) is—

$$\frac{\sqrt{\pi^6}}{\sqrt{\{(aa)(bb.1)(cc.2)(dd.3)(ee.4)(ff.5)\}}}\,e^{-(nn.6)} = \frac{\sqrt{\pi^6}}{\sqrt{\nabla}}\,e^{-(nn.6)}$$

∇ denoting, as before, the determinant whose principal diagonal is $a\,a$, $b\,b$, $c\,c$, $d\,d$, $e\,e$, $f\,f$, and the value of $n\,n\,.\,6$ being given in (iii). If we form from Ω a function Ω' by changing the sign of $n\,n$ and replacing the coefficients of the linear terms by the same quantities multiplied by $\sqrt{(-1)}$, viz. by replacing $(a\,n)$ by $a\,n\,\sqrt{(-1)}$ &c. and denote by Ω'' what Ω' becomes when the sign of $\sqrt{(-1)}$ is changed, then, u denoting the quadratic terms in Ω, we easily see that the integral considered by Todhunter (or rather that integral multiplied by the constant factor e^{-nn}), viz.—

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots\ e^{-u-nn}\cos(an.x+bn.y+\ldots)\,dx\,dy\ldots$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots\ (e^{-\Omega'}+e^{-\Omega''})\,dx\,dy\ldots$$

$$= \frac{\sqrt{\pi^6}}{\sqrt{\nabla}}\,e^{nn.6} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v)}$$

for the constituents that form the border of the numerator of $nn \,.\, 6$ in (iii) each become multiplied by $\sqrt{(-1)}$ for Ω' and by $-\sqrt{(-1)}$ for Ω'', while nn is multiplied by -1, so that the only effect of the substitution of Ω' and Ω'' for Ω is to change the sign of $nn \,.\, 6$. The value (iii) is identical with that given by Cayley at the end of Todhunter's paper, except for the presence of (nn), which is there (without any loss of generality) taken as zero. The mere evaluation of the integral when Ω contains only quadratic terms is, as remarked by Cayley, best effected at once by the theory of linear transformations (see Boole, *Cambridge Mathematical Journal*, vol. iii. p. 116); but, as a general method, Gauss's could scarcely be improved.

§ 9. In § 3 it was observed that in the case of three unknowns, x, y, z as obtained by the method of substitution required 23 tabular entries, while X, Y, Z were given by 12 more if Gauss's formulæ were used, and by 7 if Encke's method was followed. This naturally suggests an investigation of the number of tabular entries required in the different methods of obtaining the weights that are described in Encke's account in the general case when the number of unknowns is i, and such I proceed now to give. I should state that it is no more my opinion than it was Bessel's that the labour of a calculation is exactly proportional to the number of tabular entries required in its performance; but it will be generally admitted that the number of entries affords a good rough test when the methods to be compared are not unlike one another, and in the present case, the amount of work in each process is certainly measured in this way with a close approach to accuracy. The enumeration of the entries is also interesting on its own account.

Chauvenet calls all the quantities $a\,a$, $a\,b$, ... $b\,b \,.\, 1$, $b\,c \,.\, 1$ ... &c., auxiliaries; but in what follows I restrict this name to mean merely those coefficients that have to be calculated in the course of the solution of the equations (i), the coefficients of which are supposed given originally; so that $a\,a$, $a\,b$, $a\,c$ ... are excluded, and only coefficients which contain a postfix are regarded as auxiliaries. It will be convenient to follow Encke and suppose there are only six unknowns, x, y, z, w, u, t—the leading astronomical case; but the reasoning in all cases will be perfectly general, and apply when the number of unknowns is i. It will be assumed, except where otherwise stated, that the fundamental order of the letters on which the meanings of the auxiliaries depend is that of the alphabet.

Consider, then, any auxiliary, say $ef \,.\, 4$; it is defined by the equation

$$ef \,.\, 4 = ef \,.\, 3 - \frac{(ed \,.\, 3)\,(fd \,.\, 3)}{dd \,.\, 3}$$

and is therefore dependent on the four auxiliaries $ef \,.\, 3$, $ed \,.\, 3$, $fd \,.\, 3$, $dd \,.\, 3$. In order, therefore, to calculate $ef \,.\, 4$ we need the logarithms of three auxiliaries, viz., of $ed \,.\, 3$, $fd \,.\, 3$, ${}_d d \,.\, 3$; and,

besides, one antilogarithm is required in finding the value of the second term of $ef.4$. As everything is expressed in terms of the auxiliaries, it is plain that (with the exception of the antilogarithms that give the actual values of the unknowns) no antilogarithm can be required, except in the calculation of the second term of an auxiliary, and that the calculation of every auxiliary must require one antilogarithm; thus, with the exception just noticed, the number of auxiliaries is equal to the number of antilogarithms required.

§ 10. The value of t, the sixth unknown, but which is first determined, is given (*Jahrbuch*, p. 271) by the equation

$$t = -\frac{fn.5}{ff.5}$$

Consider therefore the auxiliaries and coefficients on which $ff.5$ depends, and which have to be previously obtained: we easily see, from pp. 268–270 of the *Jahrbuch*, or independently, that these are

$ff.5$

$ff.4,\ fe.4$
$\quad ee.4$

$ff.3,\ fe.3,\ fd.3$
$\quad ee.3,\ ed.3$
$\quad\quad dd.3$

$ff.2,\ fe.2,\ fd.2,\ fc.2$
$\quad ee.2,\ ed.2,\ ec.2$
$\quad\quad dd.2,\ dc.2$
$\quad\quad\quad cc.2$

$ff.1,\ fe.1,\ fd.1,\ fc.1,\ fb.1$
$\quad ee.1,\ ed.1,\ ec.1,\ eb.1$
$\quad\quad dd.1,\ dc.1,\ db.1$
$\quad\quad\quad cc.1,\ cb.1$
$\quad\quad\quad\quad bb.1$

$ff,\quad fe,\quad fd,\quad fc,\quad fb,\quad fa$
$\quad ee,\quad ed,\quad ec,\quad eb,\quad ea$
$\quad\quad dd,\quad dc,\quad db,\quad da$
$\quad\quad\quad cc,\quad cb,\quad ca$
$\quad\quad\quad\quad bb,\quad ba$
$\quad\quad\quad\quad\quad aa$

while for $fn.5$ are required, in addition,

$fn.5$
$fn.4,\ en.4$
$fn.3,\ en.3,\ dn.3$
$fn.2,\ en.2,\ dn.2,\ cn.2$
$fn.1,\ en.1,\ dn.1,\ cn.1,\ bn.1$
$fn,\quad en,\quad dn,\quad cn,\quad bn,\quad an$

In the first scheme the number of auxiliaries that occur with any given postfix is always triangular, and we see that the number of antilogarithms or auxiliaries required for the calculation of the denominator of the first unknown that is determined is

$$= 1 + 3 + 6 + 10 \ldots + \frac{i.i-1}{2} = \frac{1}{2}\left\{\frac{i^3}{3} + \frac{i^2}{2} + \frac{i}{6} - \frac{i^2}{2} - \frac{i}{2}\right\} = \frac{1}{6}(i^3 - i);$$

the numerator requires $\frac{1}{2}(i^2 - i)$ additional antilogarithms, so that the number of antilogarithms required in order to find the logarithm of the first unknown determined $= \frac{1}{6}(i^3 + 3i^2 - 4i)$ $= \frac{1}{6} i.i - 1.i + 4$, agreeing with the formula given by Encke for the number of auxiliaries (*Jahrbuch*, p. 272).

Before proceeding further, it will avoid circumlocution to denote the i unknowns in the general proposition by $x_1, x_2 \ldots x_i$, so that by the method of substitution we first determine x_i from the equation

$$x_i = -\frac{\mu n.i - 1}{\mu\mu.i - 1}$$

μ being the i^{th} letter of the alphabet, and then determine x_{i-1} from x_i and so on. Thus we shall first count the number of entries requisite to determine $\log x_i$, then one antilogarithm will give x_i, and we shall proceed to examine afterwards how many more entries are required in finding $x_{i-1}, \ldots x_1$. Sometimes the special letters and numbers having reference to the case of $i=6$ will be used, and sometimes the general symbols just mentioned, but no confusion can ever arise from this cause.

Having enumerated the antilogarithms requisite for the determination of $\log x_i$, it remains to find the number of logarithms. A little consideration shows that in the schemes written above, we require the logarithm of every auxiliary or coefficient that forms the extreme right-hand constituent of a line, and of no other constituent. The number of logarithms amounts therefore to

$$(1 + 2 + 3 \ldots + i) + i$$

(the last term being due to the auxiliaries that involve n)

$$= \frac{1}{2}(i^2 + 3i).$$

The total number of entries, therefore, that are required in finding $\log x_i$

$$= \frac{1}{6}(i^3 + 6i^2 + 5i)$$

of which $\frac{1}{2}(i^2+3i)$ are for logarithms, and $\frac{1}{6}(i^3+3i^2-4i)$ for antilogarithms.

An examination of the *Endgleichungen* (*Jahrbuch*, p. 272), shows that for the calculation of x_{i-1}, x_{i-2}, ... x_1 we require $i-2$ logarithms and $\frac{1}{2}(i^2+i)$ antilogarithms: so that the solution of the equations and determination of x_1, x_2, ... x_i, require $\frac{1}{6}(i^3+9i^2+14i-12)$ tabular entries; of which $\frac{1}{2}(i^2+5i-4)$ are for logarithms, and $\frac{1}{6}(i^3+6i^2-i)$ for antilogarithms. It is proper to remark, that this number of entries gives us $x_1, x_2, \ldots x_i$ and also $\log x_2, \ldots \log x_i$, which are required in obtaining $x_1, x_2, \ldots x_i$, but not $\log x_1$, so that if the logarithms of $x_1, x_2, \ldots x_i$ are wanted, as well as their simple values, only one fresh entry is required. From the value of nn. 6, viz. $nn - \frac{(an)^2}{aa} - \frac{(bn.1)^2}{bb.1} - \frac{(cn.2)^2}{cc.2} - \frac{(dn.3)^2}{dd.3} - \frac{(en.4)^2}{ee.4} - \frac{(fn.5)^2}{ff.5}$ (*Jahrbuch*, p. 279), we see that the calculation of (vv) only involves i additional antilogarithms, so that $x_1, x_2, \ldots x_i$ and (vv) require $\frac{1}{6}(i^3 + 9i^2 + 20i - 12)$ entries; $\log (vv)$ would require another entry.

§ 11. I now proceed to the different methods of obtaining the weights, between which the comparison really lies, as the mode of solution of the equations and calculation of (vv) is common to all the methods. The three modes of practically determining the weights which will now be considered are Gauss's (*Jahrbuch*, pp. 308, 309, and *Theoria Comb. Obs.*, § 32), and the two which are explained on pp. 313 and 314 of the *Jahrbuch*, and which, as I do not think they are given by Gauss, are probably due to Encke himself. To begin with Gauss's method, it will be seen on reference to the equations on pp. 308 and 309 of the *Jahrbuch* that, separating the numbers of logarithms and antilogarithms by a comma, $\log A'$, requires 0, 0; $\log A''$, 1, 2; $\log A'''$, 1, 3; $\log A^{iv}$, 1, 4; $\log A^{v}$, 1, 5; $\log B''$, 0, 0; $\log B'''$, 1, 2; $\log B^{iv}$, 1, 3; $\log B^{v}$, 1, 4; $\log C'''$, 0, 0; $\log C^{iv}$, 1, 2; $\log C^{v}$, 1, 3; $\log D^{iv}$, 0, 0; $\log D^{v}$, 1, 2; $\log E^{v}$, 0, 0. Thus the A's require $3+4+5$

+6 entries, the B's, 3+4+5, the C's 3+4, and the D's 3; and generally the logarithms of the factors require

$$3 + (3 + 4) \ldots + (3 + 4 \ldots + i)$$

$$= \Sigma^{*1} \frac{1}{2} (i + 3 . i - 2) + 2$$

$$= \frac{1}{2} \left\{ \frac{i^3}{3} + \frac{i^2}{2} + \frac{i}{6} + \frac{i^2}{2} + \frac{i}{2} - 6i \right\} + 2$$

$$= \frac{1}{6} (i^3 + 3i^2 - 16i + 12)$$

entries. The deduction of the logarithms of the weights from the factors requires

$$3 + 4 \ldots + (i + 1) = \frac{1}{2} (i^2 + 3i - 4)$$

entries, so that altogether the determination of the logarithms of the weights by Gauss's method requires

$$\frac{1}{6} (i^3 + 6i^2 - 7i) = \frac{1}{6} i . i - 1 . i + 7$$

entries, of which $\frac{1}{2} (i^2 - i)$ are for logarithms and $\frac{1}{6} (i^3 + 3i^2 - 4i)$ for antilogarithms. We may well close the enumeration with the determination of the logarithms of the weights, as the weights themselves are not essential. Their calculation requires $i-1$ additional entries (all for antilogarithms), in both Gauss's and Encke's first method, and $i-2$ additional entries (all for antilogarithms), in Encke's second method: log (vv) requires an additional logarithm in all three methods.

In Encke's first method the weights are determined from the following equations:—

$$ff . 5 = ff . 5$$

$$ee . 5 = (ee . 4) \frac{ff . 5}{ff . 4}$$

$$dd . 5 = (dd . 3) \frac{ee . 4}{ee . 3} \cdot \frac{ff . 5}{(ff . 4)_d}$$

$$cc . 5 = (cc . 2) \frac{dd . 3}{dd . 2} \cdot \frac{ee . 4}{(ee . 3)_c} \cdot \frac{ff . 5}{(ff . 4)_c}$$

$$bb . 5 = (bb . 1) \frac{cc . 2}{cc . 1} \cdot \frac{dd . 3}{(dd . 2)_b} \cdot \frac{ee . 4}{(ee . 3)_b} \cdot \frac{ff . 5}{(ff . 4)_b}$$

$$aa . 5 = (aa\ ) \frac{bb . 1}{bb} \cdot \frac{cc . 2}{(cc . 1)_a} \cdot \frac{dd . 3}{(dd . 2)_a} \cdot \frac{ee . 4}{(ee . 3)_a} \cdot \frac{ff . 5}{(ff . 4)_a}$$

* Σ here and in the rest of the paper means summation with regard to i form $i = 1$ to $i = i$.

wherein the literal suffixes indicate that the fundamental order of the letters is supposed to be changed, and point out which letter is removed from its original place and put last; thus $(ee.3)_c$ means $ee.3$ when the order of the letters is a, b, d, e, f, c. Whenever no literal suffix is added, the order is the fundamental order a, b, c, d, e, f. We notice that when any letter is transferred from its place to the end, the significations of all those auxiliaries that have a postfix greater than that corresponding to the letter shifted are altered; thus when c is put last, no change is made in the auxiliaries with postfixes 1 and 2, but all having higher postfixes receive different meanings. Thus, in counting the new auxiliaries we leave off, in the schemes written below, just before the line containing those postfixes that correspond to the letter which has been placed last.

Construct the following schemes:—

(e last)

$$ff.4$$

(d last)

$$\begin{array}{ll} & ff.4 \\ ee.3; & ff.3,\ fe.3 \end{array}$$

(c last)

$$\begin{array}{lll} & & ff.4, \\ & ee.3 & ff.3,\ fe.3, \\ dd.2; & ee.2,\ ed.2; & ff.2,\ fe.2,\ fd.2 \end{array}$$

(b last)

$$\begin{array}{lll} & & ee.3, \\ & dd.2, & ee.2,\ ed.2, \\ cc.1; & dd.1,\ dc.1; & ee.1,\ ed.1,\ ec.1 \end{array}$$

$$\begin{array}{l} ff.4, \\ ff.3,\ fe.3, \\ ff.2,\ fe.2,\ fd.2, \\ ff.1,\ fe.1,\ fd.1,\ fc.1 \end{array}$$

in which the law of formation is sufficiently evident to render it unnecessary to write down the scheme for a last. It will be seen that these triangular groups (omitting the bottom line of each) include all the new auxiliaries that are required; by "new auxiliaries" being meant auxiliaries which have a different meaning from what they had originally when the order was a, b, c, d, e, f, and which in the formulæ for the weights have literal suffixes. For suppose c put last so that the order is a, b, d, e, f, c, then to find what auxiliaries $fe.3$ (say) immediately depends upon, we diminish the postfix by 1, and so obtain $fe.2$, and then for the second term we have $fd.2$, and $ed.2$, besides

the denominator $dd..2$, d being the letter corresponding to the postfix 2; we therefore retain $fe.2$ and $fd.2$, but $ed.2$ has been already counted in the scheme of which $ee.3$, is the final auxiliary; and $dd.2$ has also been counted. Thus taking the letters (and postfixes) always in order, we see that in counting the number of new auxiliaries required for $ff.4$, that have not previously been counted, we retain only those that involve an f as one of their letters, and any other auxiliary that arises must also have arisen previously. The reason for the schemes is thus apparent, and we remark that the bottom line of each triangular group consists of auxiliaries that were previously required in the process of solution for $x, y, z \ldots$, so that the number of new auxiliaries, which is equal to the number of new antilogarithms,

$$= 1 + (1+3) + (1+3+6) \ldots + \left(1+3+6 \ldots + \frac{i-2.i-1}{2}\right)$$

$$= 1 + (1+3) \ldots + \frac{1}{6}(i^3-3i^2+2i) = \frac{1}{6}\Sigma(i^3-3i^2+2i)$$

$$= \frac{1}{24}(i^4-2i^3-i^2+2i) = \frac{1}{24}\, i.i-1.i-2.i+1$$

With regard to the number of logarithms, a little consideration shows that only the logarithms of the extreme right-hand auxiliaries in all the horizontal rows (including the bottom rows) of the triangular groups are required; so that the number of logarithms

$$= 1 + (1+2) + (1+2+3) \ldots + (1+2 \ldots +i-1)$$

$$= \frac{1}{2}\Sigma(i^2-i) = \frac{1}{6}(i^3-i) = \frac{1}{6}\, i.i-1.i+1$$

and the whole number of additional entries required in the calculation of the logarithms of the weights

$$= \frac{1}{24}(i^4-2i^3-i^2+2i) + \frac{1}{6}(i^3-i)$$

$$= \frac{1}{24}(i^4-2i^3-i^2-2i) = \frac{1}{24}\, i.i-1.i+1.i+2 \quad \ldots\ldots\ldots \text{(vi)}$$

The above is here called Encke's first method; but in reality he scarcely proposes it as a separate process that ought to be pursued, but only regards it as subsidiary to the method (here called his second) which he recommends, and to which I now proceed. It is, however, convenient to regard his investigation as two methods, as the one here called the first is that which was found so successful in the case of three unknowns (*see* § 3).

In Encke's second method the first three weights $ff.5$, $ee.5$, $dd.5$, are to be calculated as just described in the first method;

but the other three are to be obtained by beginning the whole process *de novo*, and solving the equations afresh with reversed order of letters, viz., f, e, d, c, b, a. We have thus a complete verification of the values found for x, y, z ... ; and the last three weights are obtained exactly as the first three were.

First, considering only the solution of the equations, it appears that only one entry, viz., that for the logarithm of fa is common to the two processes (so that the verification is very complete) ; thus the re-determination of the values of the unknowns and of $(v\,v)$ requires

$$\frac{1}{6}\,(i^3+9\,i^2+20\,i-18)$$

entries. Then if i be even, it will be easily seen that the number of entries required in the determination of the weights is found by writing $\frac{1}{2}\,i$ for i in (vi) and doubling the result, so that the number of entries

$$=\frac{1}{12}\left(\frac{i^4}{16}+\frac{i^3}{4}-\frac{i^2}{4}-i\right)$$

If i be uneven the number of entries is found by substituting $\frac{1}{2}\,(i-1)$ and $\frac{1}{2}\,(i+1)$ for i in (vi) and adding the results, we find the number

$$=\frac{1}{192}\,(i^4+4\,i^3+2\,i^2-4\,i-3)$$

A very little consideration is required in order to satisfy oneself that in the determination of the logarithms of the weights, none of the entries are common to the formulæ in which a and f are respectively the first letters ; perhaps this is best seen by fixing the attention on the denominators of the auxiliaries (*see* § 6). Thus any auxiliary with postfix 4 (say) has for its denominator the determinant whose principal diagonal is $a\,a$, $\beta\,\beta$, $\gamma\,\gamma$, $\delta\,\delta$; a, β, γ, δ being the first four letters in the assumed order ; so that no auxiliary in which the order is a, b, c, d, e, f can coincide with one in which it is f, e, d, c, b, a.

The difficulty of the enumeration consists rather in seeing that no entries are counted twice, than that none are passed over. I have given the explanation of the processes of the enumeration in sufficient detail to make them easy to follow, so that in case an error should have been made, its correction may be easy.

The general results that have been obtained are, (1.) that, i being the number of unknowns x_1, x_2, . . . x_i, the solution of the equations and the determination of $(v\,v)$ (but not of $\log v\,v$) require

$$\frac{1}{6}\,(i^3+9\,i^2+20\,i-12)$$

entries ; and that we have also incidentally $\log x_2$, $\log x_3$ . . .

log x_i, but not log x_1; (2.) that Gauss's mode of determining the logarithms of the weights requires

$$\frac{1}{6}(i^3+6i^2-7i) \quad \ldots\ldots\ldots \text{(vii)}$$

entries; (3.) that Encke's first method of determining the same requires

$$\frac{1}{24}(i^3+2i^3-i^2-2i) \quad \ldots\ldots\ldots \text{(viii)}$$

entries; and (4.) that his second method requires

$$\frac{1}{6}(i^3+9i^2+20i-18)+\frac{1}{12}\left(\frac{i^4}{16}+\frac{i^3}{4}-\frac{i^2}{4}-i\right)$$

or

$$\frac{1}{6}(i^3+9i^2+20i-18)+\frac{1}{192}(i^4+4i^3+2i^2-4i-3) \quad \ldots\ldots \text{(ix)}$$

entries, according as i is even or uneven; but that this last method gives us also a re-calculation of $x_1, x_2, \ldots x_i$ and (vv). If the weights be included as well as their logarithms, $i-1$ must be added to (vii) and (viii) and $i-2$ to (ix). From the formulæ (vii), (viii), (ix), the following Table was constructed:

i	Number of entries required in determining				Total number of entries		
	$x_1, x_2, \ldots x_i$ and (vv),	the logarithms of the weights,					
		by Gauss.	by Encke, I.	by Encke, II.	by Gauss.	by Encke, I.	by Encke, II.
3	26	10	5	26	36	31	52
4	46	22	15	47	68	61	93
5	73	40	35	78	113	108	151
6	108	65	70	117	173	178	225
7	152	98	126	171	250	278	323
8	206	140	210	235	346	416	441
9	271	192	330	320	463	601	591
10	348	255	495	417	603	843	765
11	438	330	715	542	768	1153	980
12	542	418	1001	681	960	1543	1223

It will be thus seen, that though Encke's first method requires less tabular entries for $i=3, 4, 5$, yet Gauss's is far preferable for higher values of i, and this advantage of course increases rapidly as his formula involves i^3, while Encke's has a term in i^4. Encke's second method has the great merit of affording a complete verification for everything except the determination of the

logarithms of the weights. It is only fair to notice expressly that Encke's remarks are no doubt only intended to apply to the case of $i =$, or less than, 6; and most likely he did not at all consider the relative advantages of the methods when i exceeded this limit, as such cases were scarcely likely to occur often in astronomical or other applications. If i were large, it might be worth while to extend Encke's second method and take the letters in a third order, so as to obtain a triple determination of $x_1, x_2, \ldots x_i$ and $(v\,v)$, to serve as a verification, not of the values of the unknowns, but of some of the fresh auxiliaries. Where the methods are as similar as those which have just been compared, the number of tabular entries appears to afford a very good test of the amount of work required in the performance of each.

§ 12. As an immediate corollary to the enumeration at the beginning of § 10, I add the number of entries required in the calculation, by the method of substitution, of a symmetrical determinant of i rows:

$$\begin{vmatrix} a\,a, & a\,b, & a\,c, & \ldots \\ b\,a, & b\,b, & b\,c, & \ldots \\ c\,a, & c\,b, & c\,c, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

which $= (a\,a)\,(b\,b\,.\,1)\,(c\,c\,.\,2)\ldots$

An examination of the schemes in § 10 readily shows tha the number of antilogarithms required is

$$1 + 3 + 6 \ldots + \frac{i\,.\,i-1}{2} = \frac{1}{6}\,(i^3 - i)$$

and the number of logarithms $= \frac{1}{2}\,(i^2 + i)$; to these must be added the final antilogarithm, so that the total number of tabular entries required is

$$\frac{1}{6}\,(i^3 + 3\,i^2 + 2\,i + 6) = \frac{1}{6}\,(i+3)\,(i^2+2)$$

It is very likely that the method of substitution affords the most simple mode of calculating a symmetrical determinant, and that the number of entries could not be conveniently reduced below that just found.*

§ 13. In what precedes the equations (i) to be solved have been supposed to be given, and we have been in no way con-

* I asked Professor Cayley if he knew an expression for the number of terms in the development of a symmetrical determinant, in order that I might compare the number of entries required in the above procedure with what would be necessary if the expanded determinant were calculated directly, and he in consequence investigated the very elegant formula contained in the paper published in the April number, pp. 303–307. The number of terms in the expanded determinants of the third, fourth, fifth, and sixth orders being respectively, as found by Professor Cayley, 5, 17, 73, 388, it follows that the number of entries required would be 11, 27, 88, 409, while the substitution method requires only 11, 21, 36, 57.

cerned with the manner in which they were obtained; but I cannot close this paper without alluding to the very elegant method proposed by Bessel for the formation of the coefficients $(a\,a)$, $(a\,b)$, ... by a process not unlike that of multiplication by quarter-squares. In No. 399 of the *Astr. Nach.* (vol. xvii. May 7, 1840), Bessel points out how, if there are μ equations of condition and i unknowns (and if the coefficients and not their logarithms be given), by the use of a table of squares, instead of a table of logarithms, $\mu\ (i + 1)$ tabular entries may be saved; for, calculate by the aid of the table of squares $(a\,a)$, $(b\,b)$, $(c\,c)$... then, say, $(a\,b) = \frac{1}{2}\left\{(a + b.a + b) - (a\,a) - (b\,b)\right\}$ and is given by the μ entries required for $(a + b.a + b)$, the same number as $(a\,b)$ would require if $\log a$, $\log b$ .... were known; but $(a\,a)$, $(b\,b)$ ... require only $\mu\ (i+1)$ entries, the same as would be requisite for $\log a$, $\log b$ ... so that on the whole the $\mu\ (i + 1)$ (antilogarithmic) entries that the calculation of $(a\,a)$, $(b\,b)$ ... by logarithms would require are saved. A calculation performed by n-figure logarithms is generally (nearly) correct to n figures, the only way in which figures get lost being when two terms, in both of which the first figure is of the same real and local value, have to be subtracted the one from the other, when the result is true to one figure less than either. This is often the cause of great inconvenience in logarithmic calculations, but Bessel's method by squares is liable to a much greater drawback, for in obtaining $a\,b$ from the formula $\frac{1}{2}\left\{(a + b)^2 - a^2 - b^2\right\}$ if a be large and b be small (so that the product is small), we obtain scarcely any more accuracy than if b had been large too, as we lose figures by the subtraction. Bessel noticed this himself, as he lays stress on the fact that the coefficients should not differ much among themselves, and points out how the units may be altered so as to make them more equal; but a good deal of the advantage of the method is lost if the equations have to be previously prepared. The great merit that multiplication by means of quarter squares possesses over multiplication by logarithms is, that the results are accurate, and not approximate; but here, where the table of squares would only extend to a certain number of figures, this advantage is lost. Bessel used only a small table giving 2,000 squares; but a fairly large table for the same purpose is added by Faà de Bruno at the end of his *Calcul des Erreurs*. The products of every pair of coefficients in all the equations of condition have to be found, so that one summation-product contains as a term the product of the greatest and least coefficients; and here the loss of figures might be serious: and what is worse, this loss would be concealed in the method of calculation, so that it would only be apparent when sought out, and the term in question examined separately. But theoretically the method is very pretty, and Bessel speaks highly of it as the result of practical experience.

Addition to the Paper on the Number of distinct Terms in a Symmetrical, or partially Symmetrical Determinant.

By Professor Cayley.

The generating function

$$u_0 = 1 + u_1 x \ldots + u_n \frac{x^n}{1.2 \ldots n} + \ldots, = \frac{e^{\frac{1}{2}x + \frac{1}{4}x^2}}{\sqrt{1-x}}$$

was obtained as the solution of the differential equation

$$2\frac{du}{dx} = u\left(1 + x + \frac{1}{1-x}\right).$$

Writing this in the form

$$2(1-x)\frac{du}{dx} = n(2-x^2),$$

we at once obtain from u_n the equation of differences,

$$u_n = n u_{n-1} - \tfrac{1}{2}(n-1)(n-2)\, u_{n-3};$$

and it thus appears that the values of u_n (number of distinct terms in a symmetrical determination of the order n) can be calculated the one from the other by the process

$n = 1$, $\quad 1 = 1 \cdot 1$

$= 2$, $\quad 2 = 2 \cdot 1$

$= 3$, $\quad 5 = 3 \cdot 2 - 1.1$

$= 4$, $\quad 17 = 4 \cdot 5 - 3.1$

$= 5$, $\quad 73 = 5 \cdot 17 - 6.2$

$= 6$, $\quad 388 = 6 \cdot 73 - 10.5$

&c.

which is one of extreme facility.

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*Continued Observations of the Companion of Procyon.**

By Herr Struve.

(*Translation.*)

The months of February and March were this year singularly unfavourable for Astronomical observations. In the few hours which were free from clouds, the images were for the most part so tremulous, that micrometrical measurements even of objects of comparatively little difficulty could scarcely be undertaken. Not until the 21st of March did I succeed in obtaining

* The substance of this paper was delivered orally by M. Struve, at the evening meeting, on May 8. The paper, which is written in German, has been kndly translated by W. T. Lynn, Esq., B.A.—[Ed.]

a tolerably satisfactory measure of the companion of *Sirius*. Although on that evening the images were still not of the best, yet I could not neglect the opportunity of at once examining *Procyon* in reference to the small point of light which I had remarked in its neighbourhood last year. On looking into the telescope accordingly on that night, I thought that I recognised by glimpses such an object with some certainty, of which I estimated the position-angle at about 95°; and my assistant, Herr Lindemann, succeeded in making a determination agreeing within 0°·1 with my approximate one, unknown to him when making his. Immediately after this the images became so tremulous again, that the point of light could no longer be perceived. On the 1st of April the images were better, but streaks of nebulous cloud made continuous observation impossible. During some short intervals free from cloud, we succeeded in making only three separate hasty measures of the position-angle, the mean of which gives 93°·3: a determination to which, on account of the haste with which it was necessarily made, I should only attribute a small degree of accuracy.

The following week the sky was again constantly clouded. Not till the 9th of April could I obtain the first complete measurement; after which three others, on April 13 to 15, were made, under very good atmospherical circumstances. The observations were all made with the Power III. of 309 times. Each position-angle below depends upon three determinations for the distances; the number of measures is affixed to each in parentheses.

Date.	Time of Obs.	Distance and	No. of Measures.	Position of Instrument.	Time of Obs.	Position-Angle.	Position of Instrument.
	h m	″			h m	°	
1874 April 9	9 40	10·47	(2)	I.	9 5	97·0	I.
					9 30	94·8	I.
April 13	9 35	12·17	(3)	I.	9 28	96·6	I.
					9 40	94·5	I.
					9 54	98·0	II.
April 14	9 50	12·04	(2)	II.	9 38	94·9	II.
					10 1	99·4	II.
April 15	9 40	12·00	(3)	I.	9 33	92·8	I.
					9 46	97·9	I.

Besides these, Herr Lindemann made the following measures of the angle of position:—

	°
April 9	99·3
„ 13	94·8
„ 15	96·6
Mean	96·9

In regard to the measures of distance, the agreement on the three last evenings has been remarkably favoured by fortuitous circumstances, for the separate determination on those evenings differ from each other by a whole second and more, as might indeed be expected from the difficulty of the object. It is not, therefore, surprising that the distance on the 9th of April should be smaller by 1″·70. Last year the differences on separate evenings were considerably greater.

To my observations the systematic corrections derived from measures with artificial double stars must be applied. These are for the distances quite insensible, since they amount in the case before us to only 0″·01 at the maximum. For the position-angles, on the other hand, they are considerable. Subject to small variations for the different hour-angles, they amount at the mean to 2°·95.

I would here remark, that in my paper of last year on the amount of the systematic correction, an error has been overlooked. It amounts for last year in the mean for Power III. to + 2°·66 and for Power IV. to 1°·61.

Now, if we take the mean values from my observations for the two years, we have

1873	March 28	$d = 12''.49$	$P = 87°.65$
1874	April 10	11·67	96·65

or, after applying the systematic corrections:

1873	March 28	$d = 12''.49$	$P = 90°.24$
1874	April 10	11·67	99·60

The distance in the interval would, therefore, seem to have diminished by about 0″·8. But owing to the difficulty of the measures, the mean values themselves must be subject to such uncertainties that the reality of the apparent diminution must be considered doubtful. But in regard to the increase of the position-angle, there can scarcely be any doubt. Not only does the observed increase of 9°·5 correspond to a considerable linear change of place, amounting to 2″·0, but the measures of direction are in themselves much easier, and more certain than those of distance.

It is well known that Professor Auwers, as soon as he had received my observations of last year, repeated his investigations into the variable proper motion of *Procyon*, availing himself also of the observations of this star which has been made since 1862. From this he concluded that it was doubtful whether the object observed by me was really the sole body disturbing the proper motion of *Procyon*, but that the doubt would be removed if it appeared this spring that the position-angle had undergone an increase of from 9° to 10°. And this increase has really shown itself above in the most remarkable manner. I consider it,

therefore, to be decisively established that the object I have observed is actually the companion whose existence has been theoretically proved by the calculations of Auwers; and hope that the astronomical world will rejoice with me in the triumph thus obtained for the labours of my honoured friend, and through them for our common science. In order to remove any exception that might be taken that the wished-for result had in any degree been itself the cause of the recognition, and affected the measurement of the place of so difficult an object, I will just remark, that I had not looked again at the paper of Auwers in question since its first receipt last summer, and had totally forgotten the data of its criterion, and the mutual relation of the two stars. I did not again take it up until after I had succeeded in making the first observation, and the results of that paper were even less present to the mind of my assistant, Herr Lindemann, whose younger eye appears generally to have seen the companion even better than mine.

On the other hand, we cannot lose sight of the important fact that various other astronomers, using very powerful instruments last winter and spring, have, so far as has hitherto been known to me, sought in vain for the faint companion. In particular, this has been the case with the new refractor of the Washington Observatory of 26 inches aperture, with the performance of which Professor Newcomb expresses himself as well satisfied. Of the cause of the failure I can, not being in possession of detailed information concerning the circumstances of the observations, merely form conjectures. Perhaps the atmospheric conditions during the Washington observations were less favourable than those here; perhaps also the surpassing brilliancy of the principal star may have hindered the recognition of the small companion in its neighbourhood, if the observer made use of the whole aperture of 26 inches. How great is the effect of the brilliancy of the principal star has shown itself very clearly in our observations. Although the near companion is doubtless considerably brighter than the known little star, which is near *Procyon* without belonging to its system, and precedes it about $42''$ to the north, the latter could always be seen in a dark night with ease by all observers who could not see the former at all, which was best seen in feeble twilight, whilst the more distant little star was not visible at all. But perhaps, also, practice in the recognition of such objects, and the greater or less sensibility of the retina, may have had their influence. In this respect also our observations have furnished some remarkable facts. In the last favourable observing evenings I requested six other astronomers to look successively at the interesting object. But whilst Herr Lindemann and myself saw and measured it with comparative ease; of the six in question only one—Herr Ceraski, second Assistant at the Moscow Observatory—succeeded in seeing it with certainty without being warned of the place where it was to be looked for. Of the others some thought they perceived it after

its place had been accurately pointed out to them, but not with satisfactory certainty. I myself always made the observations best if I came to the telescope with fresh eye. After some minutes the eye became usually so exhausted that I was obliged to give it some rest to enable me to see the object. Some of the later observations on separate evenings are therefore of inferior accuracy.

The difficulty of the observation of the companion of *Procyon* will of course be unequally greater than that of the companion of *Sirius*. In our latitudes this difficulty is somewhat equalised by the low position of *Sirius*: for the companion of which I have looked in vain on many evenings in the years immediately after its discovery by Alvan Clark, until at last favourable atmospheric conditions enabled me to see it distinctly. But after I had once recognised it, and thereby followed the modifications of its appearance with tremulous images, it is now only when there is a high degree of tremor that I cannot see it, at least by glimpses, with certainty. These experiences have doubtless been of use to me in the recognition of the companion of *Procyon*, and give me also full confidence in the accuracy of my perceptions, independently of the corroborating observations of Lindemann and Ceraski. Notwithstanding this, however, I willingly acknowledge that a confirmation of them by other observers at other places with other instruments, would be very welcome to me.

On a Method of finding the Parallax of Double Stars, and on the Displacement of the Lines in the Spectrum of a Planet.

By Professor C. Niven.

(*Communicated by W. H. M. Christie, Esq.*)

§ 1. The idea of Doppler that the motion of the fixed stars towards or from the Earth must exercise an influence on the nature of the light which we receive from them, though it has not served his purpose of explaining their colours, has yet not been without fruit. For although, in general, no great change of colour can be looked for in this direction, we may nevertheless expect to find traces of the effect in a displacement of the fixed lines of the spectrum. The displacement of the line F of hydrogen for a velocity of 20 miles per second amounts to nearly $\frac{1}{12}$th of the distance between the constituents of the double line of sodium, and is towards the violet or red end, according as the star is approaching or receding from us. Now displacements of this magnitude are fully within the reach of modern instruments, and accordingly Mr. Huggins has succeeded in estimating the velocities of separation or approach of a number of the brighter stars.

At the meeting of the British Association in Edinburgh, Mr. Fox Talbot suggested the application of the spectroscopic method to the determination of the distances of certain double stars. It is manifest that if we know the form of the orbit of a binary system, and its position with regard to the plane of the heavens, or in other words, of its apparent orbit, we require only one linear element to determine its absolute magnitude. And if this be given, the parallax of the system can be readily found; and, moreover, if we suppose, as there is every reason to do, that the law of gravitation holds good for such star systems, we have the means of determining the sum of their masses.

The linear element required would be supplied by an observation of the component in any direction of the relative velocity of the two members of the system, as, for instance, the component along the line joining the Earth with the star. For this depends on the orbital motion only, and not on any proper motion which the system as a whole may possess.

The difficulties in the way of effecting this determination arise from three sources:—the faintness of the stars, the smallness of their angular separation, and possibly also from the component of velocity under consideration being too small to be capable of measurement. With regard to the first two of these, it may be observed that Mr. Huggins has recorded in the *Philosophical Transactions* for 1864, that he was able to examine separately the two components of *β Cygni*, and of *α Herculis*. Now the magnitudes of the two components of the former of these are respectively 3 and 7 (Smyth), while those of the latter are $3\frac{1}{2}$ and $5\frac{1}{2}$, and are separated by an angle of $4''{\cdot}6$. It is therefore inferior in both respects to *Castor*, the two members of which are of magnitudes 3 and $3\frac{1}{2}$, and which were separated at the beginning of the year 1869 by a distance of nearly $6''$.

With regard to the third difficulty, it is only in one case, that of *α Centauri*, that we can speak with any certainty. Adopting the parallax of $0''{\cdot}913$, and the old elements calculated by Captain Jacob, I find the greatest velocity of approach to amount to 12·556 miles, and the greatest velocity of separation to be about 25·85 miles per second. With the elements more recently given by Mr. Powell, the greatest velocities of approach and separation are respectively only 6·82 and 20·41 miles; while about the present epoch the velocity is one of separation amounting to about 17·43 miles per second.

Under these circumstances there seems good ground for believing that the observations required are not beyond the reach of the present instrumental means; I have therefore thought that some service might be done by removing the geometrical obscurities of the problem, and showing what the formulæ are which serve, by the aid of the observations, to determine the parallax.

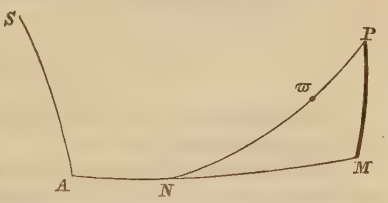
The general result of the investigation is a relation of the

form $p\,W = A\cos\theta + B$; wherein θ is the position of the secondary measured on the relative orbit from the node, p the parallax of the system (in seconds), W the component of the relative velocity measured *from* the Earth along the line to the system, and A and B two constants to be calculated in each case from the elements of the orbit. Here, as throughout the present paper, W is taken to be positive when considered as a velocity of separation, and negative when one of approach.

Some general deductions, which can be made from the above formulæ, will be considered in their proper place.

In Articles 5 and 6 I have considered the effect which the motion of a planet has on the lines in the spectrum of the light reflected from it.

§ 2. In considering the motion of a binary star system we take for plane of reference the tangent plane to the heavens, for the initial line that in which this plane is cut by a declination circle, and the direction of measurement to be north—following —south—preceding. It seems sufficiently established that the relative motion of the system may be represented by supposing the secondary component to trace out an ellipse in a plane inclined to the plane of reference conformably with the ordinary laws of elliptic motion, and having the primary as focus.



Let S be the primary, A the origin of measurement, N the node of the orbit, ϖ periastre, and P any position of the secondary; let, also, the arc P M be drawn at right angles to the plane of reference. Denote as usual the elements of the actual ellipse by a, e, N, i, ϖ, and by T the time of revolution in years. Let $N\,P = \theta$, $N\,M = \theta$, and $P\,M = \lambda$.

We thus have, first of all,

$$\left.\begin{aligned} \text{angle of position} &= N + \theta_1 \\ \tan\theta &= \tan\theta_1 \sec i \\ \sin\lambda &= \sin i \sin\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

To connect the actual and angular mean distances a and a'', we have

$$a : \nabla :: a : p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

∇ being the diameter of the Earth's orbit.

If we call $S\,P = r$, we find from the theory of elliptic motion:

$$\frac{1}{r} = \frac{\mu}{h^2}\left(1 + e\cos\overline{\theta - \varpi}\right)$$

$$\frac{dr}{dt} = \frac{\mu e}{h}\sin\overline{\theta - \varpi}$$

$$r\frac{d\theta}{dt} = \frac{\mu}{h}\left(1 + e\cos\overline{\theta - \varpi}\right).$$

The motion of the star is compounded of the velocities $\frac{dr}{dt}$ along S P and $r\frac{d\theta}{dt}$ at right angles to it. The component of the former at right angles to the plane of the apparent orbit is

$$\frac{dr}{dt}\sin\lambda, = \frac{\mu e}{h}\sin i\ \sin\theta\ \sin\overline{\theta-\varpi},$$

and that of the latter is

$$r\frac{d\theta}{dt}\sin i\cos\theta.$$

Thus, the whole component of velocity, W

$$= +\frac{\mu}{h}\sin i\,(\cos\theta + e\cos\varpi).$$

Now we have also

$$\frac{\mu}{h} = \frac{2\pi a}{\mathrm{T}\sqrt{1-e^2}},$$

whence, if W be measured in miles per second, and n be the number of seconds in a year,

$$\mathrm{W}.\mathrm{T}.n = \frac{2\pi a\sin i.\nabla}{p\sqrt{1-e^2}}(\cos\theta + e\cos\varpi) \quad . \quad . \quad . \quad . \quad (3)$$

[It is assumed in this result that the motion of the secondary in its apparent orbit is *direct*.]

§ 3. It follows from the expression just found that W attains its greatest values at the two nodes, being a positive maximum at the one and a negative maximum at the other. Of the two that is the greater which corresponds to the node nearest periastre, and this will be positive or negative according as the star in passing through periastre recedes from or approaches the nearest node. At two points situated symmetrically with respect to the line of nodes W vanishes. These points separate the region of the orbit from which W is positive from that in which it is negative, and are determined by $\theta = \pm\beta$, where $\cos\beta = -e\cos\omega$.

If we write $p\ \mathrm{W} = \mathrm{A}\cos\theta + \mathrm{B}$, the two maxima are A + B and A ~ B ; B represents the mean value for equal angular spaces round the primary.

The annexed Table contains the values of the more important quantities for most of the orbits whose elements are known. Columns 5 and 6 contain the maxima values of p W, and include both magnitudes and signs. In column 8 are found the angles of position corresponding to the points for which W vanishes. The symbol + between the two angles indicates that, for positions of the secondary between the minus, W is positive. The reverse is denoted by the symbol −.

TABLE I.

Star.	Elements computed by	A.	B.	Greater max. value of p W.	Less max. value of p W.	Log A.	Angles of Position for which p W = o.	
							° ′	° ′
η Coronæ Bor. . .	Mädler . .	·92646	·04704	− ·97350	+ ·87942	$\bar{1}$·966 8285	105 22	− 303 14
ζ Cancri . . .	Mädler . .	·7821	·1281	+ ·9102	− ·65395	$\bar{1}$·893 2505	71 11½	+ 291 44½
ξ Ursæ Majoris . .	Herschel .	1·9528	·4097	+ 2·3625	− 1·5431	·290 6615	153 44	+ 41 50
ω Leonis . . .	Villarceau .	·3608	·2311	− ·5918	+ ·1297	$\bar{1}$·557 2011	174 41½	− 95 40½
p Ophiuchi . .	Herschel .	1·6851	·6502	+ 2·3353	− 1·0349	·226 6103	184 52	+ 69 58
ξ Boötis . . .	Herschel .	4·8088	2·8552	− 7·6640	+ 1·9536	·682 0362	166 58	+ 193 7
δ Cygni . . .	Hind . .	·3381	·09184	+ ·4300	− ·2463	$\bar{1}$·529 0691	91 39	+ 337 9
γ Virginis . . .	Herschel .	·7002	·4566	− 1·1568	+ ·2436	$\bar{1}$·845 2215	183 52½	+ 317 43½
Castor . . .	Herschel .	1·6903	·1669	+ 1·8572	− 1·5234	·227 9533	96 52	+ 309 18
σ Coronæ Bor. . .	Hind . .	·1624	·04147	+ ·2039	− ·1210	$\bar{1}$·210 6979	127 22½	− 274 43½
μ_2 Boötis . . .	Hind . .	·2445	·04719	+ ·2916	− ·1973	$\bar{1}$·388 1834	191 16½	+ 43 25½
α Centauri . .	Jacob . .	17·533	6·0684	+ 23·6014	− 11·4646	1·243 8514	204 57½	− 327 16½
α Centauri . .	Powell . .	12·433	6·2070	+ 18·640	− 6·226	1·094 5392	9 27½	+ 39 8½
ζ Herculis . .	Villarceau .	·9765	·1127	+ 1·08914	− ·8638	$\bar{1}$·989 6616	115 13	+ 313 29

The great preeminence of *α Centauri* in this list is due, mainly to the largeness of its mean distance caused by its proximity to us. With the aid of the commonly received value of its parallax, I have calculated the two maxima values of W and its value at the present epoch. They are given at the commencement of this paper.

§ 4. Through the kindness of Mr. Gill, who has supplied me with some of Dembowski's recent measurements of binary systems, I am enabled to add the present values of p W in certain cases. The observations do not give directly the values of the distance and position-angle at the present time; but we may approximate to them by assuming their rates of change, as determined by two sets of observations, to remain sensibly constant for a few years. In two cases the observations of Dembowski have been combined with those of Brünnow, made at Dunsink during the years 1868-69.

The distances are given in the table to indicate what stars are more favourably situated for observation :—

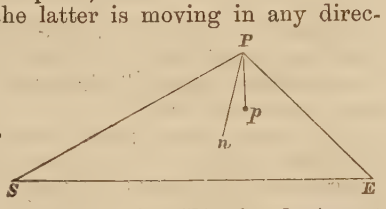
TABLE II.

Star.	Position-Angle and Distance.							Value of p W.
	Observed.				Calculated.			
	Epoch.	Position-Angle.	Distance.	Observer.	Epoch.	Position-Angle.	Distance.	
		°	″			°	″	
η Coronæ Bor.	1872·44	50·86	...	D	1874·44	61·44	0·975	+ ·3167
	1873·44	56·15						
ζ Cancri	1872·33	162·84	...	D	1873·33	135·86	1·0	+ ·4432
	1873·33	150·23	0·5					
ξ Ursæ Majoris	1872·33	19·39	...	D	1873·33	338·43	·882	+ 1·0078
	1873·33	358·91						
p Ophiuchi	1868·90	98·008	4·92	B	...	...	...	+ ·4565
δ Cygni	1868·69	349·06	1·52	B	...	...	...	+ ·09307
γ Virginis	1869·22	164·98	...	B	1873·33	159·01	4·52	− ·4757
	1872·86	160·79	...	D				
Castor	1869·16	238·97	5·71	B	...	...	...	− ·86162
σ Coronæ Bor.	1868·55	194·048	...	B	1873·33	199·33	3·13	− ·12087
	1872·96	198·08	...	D				
ζ Herculis	1872·48	173·88	...	D	1873·33	153·62	1·43	+ ·4893
	1873·52	162·49						

§ 5. *Displacement of the Lines in the Spectrum of a Planet.*—In all the cases, so far as I am aware, in which the effect of the motion of the source of light on the spectrum lines has hitherto been treated, the body has been supposed self-luminous. But it may be readily shown that somewhat similar results obtain when the substance shines either by reflected or transmitted light. To make this clear in the case of reflexion, which is the more important one from an astronomical point of view, let us first consider what would happen if the spectator and source of light were in a line with the reflecting body, and the latter were moving towards us with a velocity v. Supposing any change of phase which may occur at reflexion to be unaffected by the motion of the mirror, it follows that the path of a second luminous crest will be shorter than that of the first by twice the distance which the mirror has travelled during the period of a single undulation. The number of vibrations per second in the reflected light is therefore increased in the ratio $1 + \frac{2v}{V} : 1$, where V is the velocity of light. The wave-length of the light which the spectator receives is shortened in the same ratio.

The effect of the mirror in this case is, thus, double what would take place if the body were self-luminant.

Suppose now, in the second place, that the incidence on the reflector is oblique, and that the latter is moving in any direction in the plane of incidence. Let S be the source, E the spectator, P the reflector, P p the distance moved in time $\delta t = v\,\delta t$, P n the bisector of the angle S P E. Let also S P E $= 2\alpha$, p P $n = \beta$.



Since it is a theorem that, for the point (P) of reflexion at the mirror, S P + E P is a minimum, and therefore stationary, we may still in the new position of the mirror treat p as the point reflexion. By the motion of P the path of the light is shortened by a length $v\,\delta t\,(\cos\overline{\alpha + \beta} + \cos\overline{\alpha - \beta}) = 2v \cos\alpha \cos\beta\,\delta t$. The period of vibration and wave-length are, thus, each shortened by a fraction of itself $= \frac{2v \cos\alpha \cos\beta}{V}$. The reverse would happen if P moved in the direction p P.

[A very similar formula expresses the effect of any motion possessed by a refracting surface; and it follows that, in transmission through a plate with parallel faces, motion of the latter produces no effect whatever on the lines of the spectrum.]

§ 6. These principles may be applied to the analysis of the light which we receive from the planets. And in showing how this may be done, I shall adopt the restriction that all the planets revolve in the ecliptic, in concentric circles round the Sun.

The results which we shall thus obtain will not be strictly accurate, but will probably not differ much in their general features from those actually presented in nature. They will, at least, enable us to judge, with more or less truth, of the magnitudes of the numbers with which we have to deal.

Since the magnitudes with which we are concerned are exceedingly minute, we may employ the principle of the superposition of small changes to separate the effect due to the motion of the planet from that of the Earth. Let these bodies be represented as in the former figures by P, E, and let their distances from the Sun be a, b miles, and let their times of revolution be T and t years. P and E are moving at right angles to S P, S E with velocities $\frac{2\pi a}{\mathrm{T}n}$ and $\frac{2\pi b}{tn}$ miles per second, the direction of revolution being from E to P. Denoting these by u, v, it is evident that the joint effect of the motion of both is to increase the wave-length of the light emitted from the Sun by a fraction of itself $= \frac{\mathrm{U}}{\mathrm{V}}$, where

$$\mathrm{U} = 2u\cos\alpha\,\sin\alpha - u\,\sin \mathrm{S\,E\,P}$$

$$= \frac{2\pi}{n}\left(\frac{1}{\mathrm{T}} - \frac{1}{t}\right) b\,\sin \mathrm{S\,E\,P} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

From this formula we gather the general result that, for equal angular distances of a planet for the Sun, the values of U are numerically equal. To investigate it more narrowly, two cases must be distinguished,—

1°. For an *inferior* planet, for which $t > \mathrm{T}$, U is positive when the planet is east of the Sun, and negative when west of it. At the greatest elongations of a planet U is a maximum and numerically $= u\left(1 - \frac{\mathrm{T}}{t}\right)$.

2°. For a *superior* planet, for which $t < \mathrm{T}$, U is negative for positions east of the Sun, and positive for western positions. Its maximum is reached when the distance is a right angle, and it then amounts to $u\left(1 - \frac{t}{\mathrm{T}}\right)$.

The following Table gives the values of U_g for the principal planets calculated with the restrictions above mentioned :—

TABLE III.

Planet	Ug in miles per second
Mercury	22·24
Venus	8·24
Mars	8·57
Asteroids	13·87
Jupiter	16·76
Saturn	17·68

Queen's College, Cork,
1874, *April* 11.

On a piece of Apparatus for carrying out M. Janssen's Method of Time-Photographic Observations of the Transit of Venus. By Warren De La Rue, D.C.L., F.R.S.

(*Described verbally at the Meeting of the* 10*th of April*, 1874.)

It will be remembered that M. Janssen proposed to take advantage of the preparations which were being made for photographic observations of the transit of *Venus*, by adding thereto a special contrivance by means of which a series of photograms might be taken in rapid succession, when *Venus* had actually entered on the limb of the Sun. It was thus hoped to ensure a more exact time-record of the internal contacts of the planet and the limb of the Sun, than could be obtained by eye-observations.

The general scheme of the apparatus necessary to effect this object was well considered by M. Janssen, and there is little novelty in the form of the instrument I am about to describe; nevertheless, as there are some peculiarities in its construction, and as it contains the fewest possible number of moving parts, I have thought that it might not be unworthy of the notice of the meeting.

The photographic plate to be employed is supported in a cell which rotates on an axis excentric to the axis of the telescope;

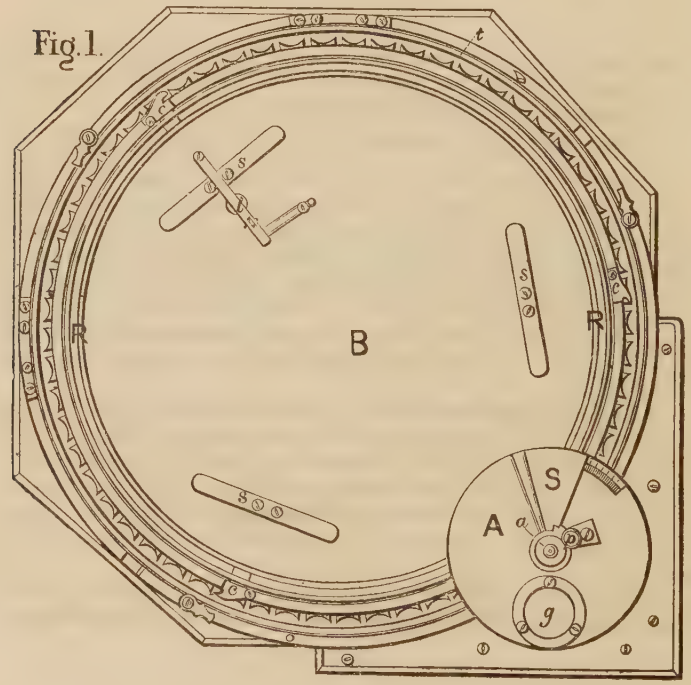
in the cover of the fixed frame in which the cell rotates is a radial slit near the centre of the telescope and exposing a portion of the zone (one-sixtieth of its circumference); just above the cover of the plate-holder is a circular revolving shutter with an adjustable radial slit, the axis of this shutter is just beyond the periphery of the cell, and is therefore excentric to the cell; it carries a pin which falls into one of sixty radial slots racked in the edge of the plate-holder and carries it round one step for every revolution it makes. This axis, during each revolution, exposes one-sixtieth of the sensitive plate, then shuts off the light, and lastly brings a fresh portion of the film into position for the succeeding photogram.

Each time the instantaneous shutter exposes the plate contact is made with a wire from a voltaic battery and an electric circuit established with the electro-magnet of a chronograph.

The sensitive plate to be used is a disk somewhat less than 11 inches in diameter; the fixed frame or casing is about 13 inches in its largest dimension and $2\frac{1}{4}$ inches thick: the whole apparatus weighs twelve pounds and a quarter including the glass disk.

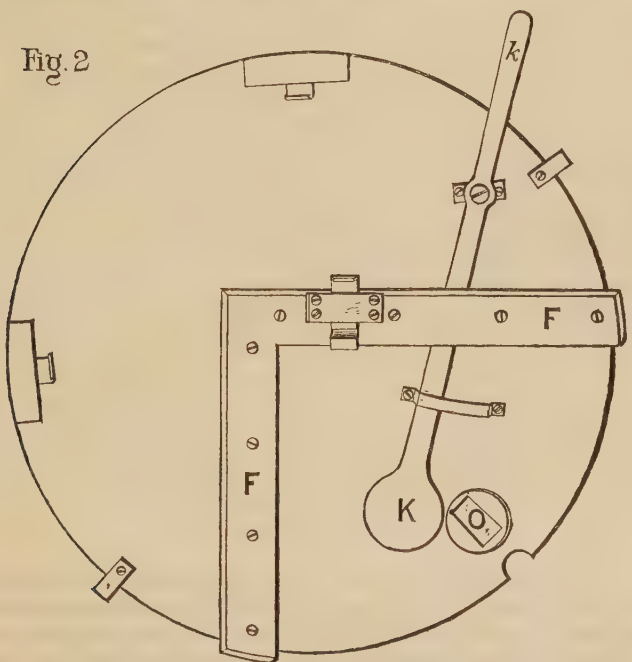
Fig. 1 shows that face of the apparatus which is turned towards the secondary magnifier when it is placed in the photoheliograph; the cover, fig. 2, having been removed to show the several parts.

Fig. 1.



B, fig. 1, is the bottom plate of the circular cell which holds the glass disk; it is racked with 60 radial slots *t* and 60 circular spaces (these are shown in full size in fig. 4), *c c c* is a ring screwed to B to form the cell, it is electro-silver plated on the inside to prevent the nitrate of silver from acting upon it if wet plates are used. A removable ring R R, also electro-plated, serves to hold the glass disk in a plane at right angles to the axis of the telescope; it can be taken out to permit of the insertion of the glass disk after turning aside the catches *c c c*. The disk may then be dropped (the sensitive film uppermost) gently into its place on three pressure springs *s s s*, and the ring R R pressed on it; the catches *c c c* when turned over the ring keep it exactly in its proper place, as they are bevelled at the edge and slide under

Fig. 2



dovetailed recesses in the rim of the cell. The cover shown in fig. 2 must then be placed over the plate and secured by the bolts and catches shown in fig. 1; the radial opening O must, of course, be covered by the shutter K before the apparatus is taken to the photoheliograph.

A, fig. 1, shows the instantaneous apparatus: it is a circular disk 4 inches in diameter, having a radial opening of about 1 inch at the periphery; the aperture is provided with a shutter by which

it may be completely closed, or the shutter may be opened to any extent by turning the pinion *p* which works into a portion of the shutter toothed for that purpose; the adjustment is facilitated by a graduated arc shown in the figure. On the opposite side there is inserted a disk of so-called ruby glass (copper-red) 1 inch in diameter; in a certain position of the cell this permits of the Sun's image being seen and adjusted to its proper place.

Fig. 3.

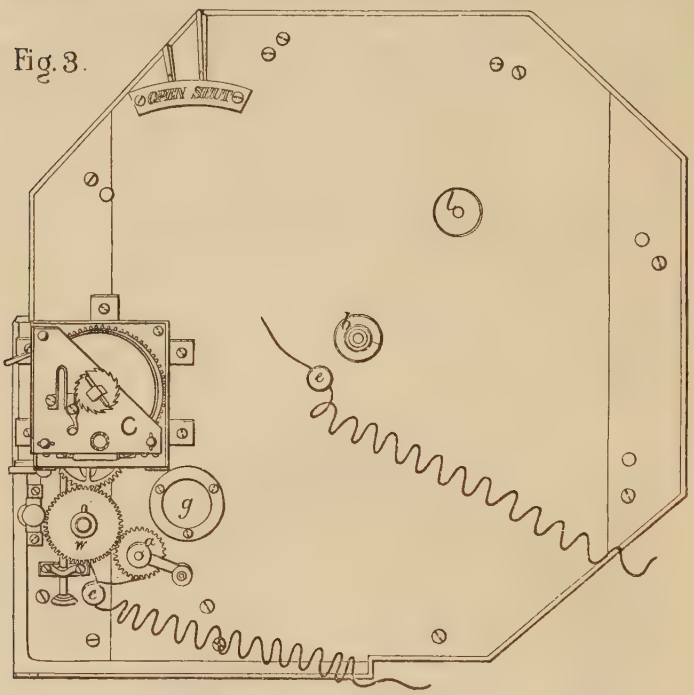


Fig. 3 represents the back of the apparatus; *b* is the axis of the plate-holder B; *a* is the axis of the instantaneous shutter; on it is fixed a toothed wheel to connect it with the clock-work driver, and also a handle to permit of its being turned by hand. *w* is an intermediate wheel always in gear with the wheel on *a*, but attached to an arm moving round *a* which permits of its being thrown in and out of gear with the driving-wheel of the clock C. In preparing the apparatus for work, *w* is thrown out of gear and the handle turned from left to right a sufficient number of revolutions to bring it to a stop, provided for that purpose; the stop is formed by racking the last of the slots *tt* for only a small distance.

The sensitive plate is now put into its cell, as before de-

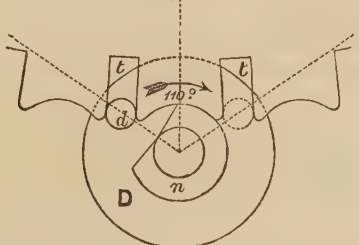
scribed, and the apparatus attached to the photoheliograph by means of the framework, F F fig. 2, fixed to the moveable cover; a screw-bolt in the photoheliograph fastens the framing firmly in its place. Attention having been paid to the directions given in the preceding paragraph, the aperture *g* covered with red glass will correspond exactly in position with similar red glass covered apertures in the instantaneous shutter and the bottom of the plate-holder B, and the Sun's image can be then seen and adjusted in position when the handle *k* of the shutter K has been set to the word "open."

The wheel *w* has then to be geared with the clock-driver C (previously wound up and stopped by a small detent, which acts against the fly governor). All being now ready, and the signal having been given, the detent, shown in fig. 3 on the left at the top of the clock-driver, is released with a slight touch of the finger and the apparatus starts and takes 60 successive photograms at any required interval; suppose at intervals of a second or of two seconds. A regulating pressure screw permits of the interval being varied within certain limits. After 60 photograms have been taken the apparatus stops because the pin *d*, fig. 4, enters the short slot.

On the axis *a* of the instantaneous shutter is fixed a disk of ivory in the periphery of which is inserted a platinum pin screwed into the pinion *a*; an insulated spring tipped with platinum presses against the edge of the ivory disk and completes the circuit of a voltaic battery connected with the chronograph at the exact instant that the slit in the shutter exposes a portion of the sensitive plate. Thus, even if the movement is not transmitted with perfect regularity, the precise epoch of each photogram will, nevertheless, be recorded.

It only now remains to describe the very simple mechanical contrivance by means of which a sixtieth part of the sensitive plate is successively brought into position under the radial

Fig. 4.



opening O, fig. 2, in the cover of the apparatus. Referring to fig. 4 drawn full size, D is a disk having a boss *n*, it is fixed on the axis *a* of the instantaneous shutter; in D is fixed a pin *d*

which, once in every revolution of the axis *a*, enters into one of the radial slots *t* of the plate-holder and carries it round one-sixtieth part of a revolution, reaching then the position shown by the dotted circle after having passed through an arc of 110°. Before it leaves the slot, the boss *n* has passed into one of the circular recesses corresponding to its own curvature, and locks the plate-holder firmly. In order to permit of the rotation of the plate-holder at the proper time, a portion of the boss *n* is cut away to a sufficient extent, as shown in the figure.

The apparatus just described has been fitted up for the Indian photoheliograph.

I may here state that Mr. Christie and M. Dallmeyer have contrived another form of apparatus also founded on M. Janssen's general proposition; and I will now describe another modification (verbally described at the meeting May 9th), which in reality is the one which I first devised. It was designed in order that it might exactly fall within the channels provided in the photoheliographs for the plate-holder to contain plates 6 in. × 6 in.; the conditions were more difficult to comply with than in the before-described plan, and the apparatus is conse-

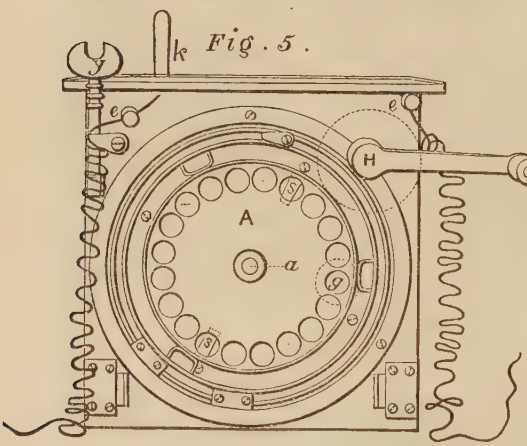


Fig. 5.

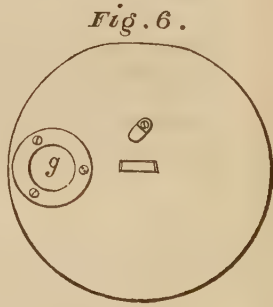


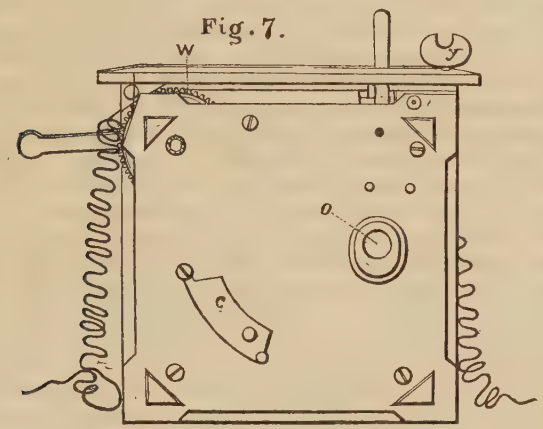
Fig. 6.

quently somewhat more complicated. Fig. 5 shows the back of the apparatus, that is the side turned away from the secondary magnifier. The cell A which holds the plate is perforated with twenty circular apertures through which the image of a portion of the Sun will be formed; it and the instantaneous apparatus rotate on the stud *a*; the sensitive film in this apparatus is placed downwards and rests on three bridle-wires of silver against which the plate is pressed by means of springs at the back of the cover fig. 6; the cover in this apparatus rotates with the cell to

which it is fastened by clips. There is a piece of red glass *g* inserted in it to permit of the Sun's image being seen; it corresponds in a certain position of the cell with similar pieces, one of which is inserted in the instantaneous shutter, the other in the shutter *k*.

The instantaneous shutter is concentric with the cell which holds the plates, and has two adjustable radial apertures *s s* Fig. 5,

Fig. 7.



which can be adjusted by means of the spanner *y*, after turning aside the door *c*; it is so geared with the wheel W on the axis of the handle H, fig. 7, that it makes half a revolution for one turn of the handle. On the axis of the handle is a pin which once in every revolution takes into one of the twenty epicycloidal teeth racked in the edge of the plate-holder, and between these teeth are circular locking recesses for a boss on the axis to enter and to lock the plate in position. *o* is an aperture which permits of the Sun's rays passing through one of the openings in the plate-holder.

To prepare the apparatus for work the handle H has to be turned from right to left a sufficient number of times to bring the driving-pin to the stop; the red glasses are then exactly one over the other. The handle must be turned in a contrary direction (left to right) to produce the photograms until after twenty pictures are obtained the pin again comes against the stop. There is a provision *e e* in the apparatus for electric communication with a chronograph. This particular apparatus is destined for the Melbourne Photoheliograph now in course of construction by M. Dallmeyer, as there is not sufficient time to provide one for sixty photograms.

On an Apparatus designed for the Photographic Record of the Transit of Venus.

By Signor J. Capello.

(*Extract from a letter to Dr. De La Rue.* Translation.)

I have just received your precious letter, with the sketch of the apparatus for M. Janssen's process. It is the only method of obtaining the exact instant of the contacts, free from personal errors, &c., in order to calculate the parallax by the methods of Halley or Delisle, and afterwards to make the comparison with the photographic methods properly so called, that is to say, the measurement of the series of photograms obtained after the first and before the last internal contacts. I have also received Mr. Abney's process: I was already acquainted with it from the *Photographic Year-book* of 1874, but I had not yet used it.

It is with the greatest regret that I have to announce to you that our expedition has failed. Our government, after having approved (May 1873) the plan and budget proposed by our Academy of Sciences: after having spent more than 350*l.* in preparations of all kinds, has recently (10 April 1874) abandoned the plan, for want of authority from the Chambers to pay the costs of the expedition: these costs do not exceed 2,000*l.*

We had made preparations and experiments for proceeding on the following plan. The difficulties were vanquished.

The dome or tower of the photoheliograph was itself a camera obscura: there was in a moveable shutter a hole always covered by the instantaneous apparatus: and another with a red glass, in order to point the photoheliograph by means of a lens at the side of the tube; this lens acting as a finder.

The instantaneous apparatus is quite independent of the photoheliograph; it is on an opaque shutter upon the moveable roof covering the hole *A* which allows the passage of the solar rays at the instant of the impression: this apparatus is composed of a cardboard disk, with a cut-out sector, figs. 1 and 2.

Fig 1.

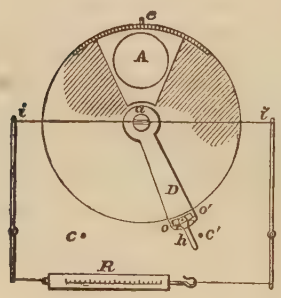
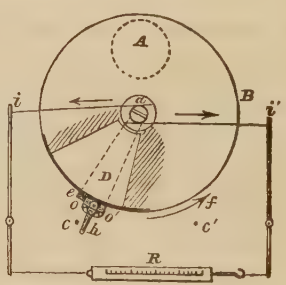


Fig. 2.



The axis (a) is pierced with a hole through which passes a thread $i\,i'$, which is kept stretched by the spring R by means of two vertical levers. In the figure 1 the apparatus is at rest; by making it assume the position of figure 2, moving in the direction $A\,B\,e$ until the tooth e catches in the click h, it is drawn forcibly back by the spring, and tends to return in the contrary direction to a point near c'. Moving the arm D to which is fixed the click h, until it presses against the fixed point c the tooth e escapes, and after making a whole revolution the tooth e again catches, but on the side c'. Making the pointer move subsequently from c towards c', and after having touched h at c', the disk describes another revolution, but in the contrary direction, and so on alternately (from left to right and from right to left).

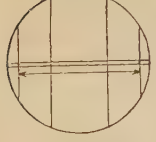
To each revolution corresponds an exposure. The duration of the exposure can be regulated, to within some thousandths of a second, to 0·4 or 0·5 of a second, according to the tension of the spring R.

The whole of this apparatus is the invention of one of the members chosen for the observations, M. Campos Rodrigues, of the Astronomical Observatory.

The dome of the photoheliograph being dark, the dry plates can be handled perfectly without the aid of the plate-holder (slide): the plates resting on three silver points by means of a simple system of supports, also contrived by M. Campos Rodrigues. By this method one can, without difficulty, expose three plates in every two minutes, or even more rapidly.

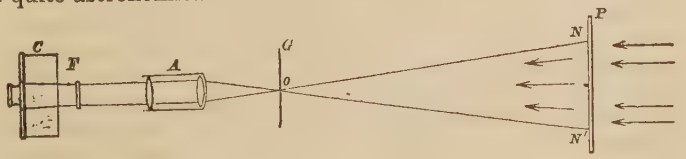
The dry collodion we had fixed upon was that chosen by the German Commission, and made by Dr. Fogel of Berlin. The process is Fothergill's albumen-process, with slight modifications; it has given me very good images of the Sun; and according to Vogel's experiments, verified by myself, it contracts very slightly.

Fig. 3.



We have had spider threads on a very exact micrometer placed in the focal plane of the object-glass: the two horizontal ones being in the direction of the celestial equator, and the others perpendicular to it. The distance between the exterior ones carefully measured by the times of star transits would be our unity for measurement of the photograms.

The distortion always met with in the secondary magnifiers would be quite eliminated by the method of measuring the images of the photograms, due to M. Campos Rodrigues. The principle is quite astronomical.



In the measurements we employ the same magnifier A, and in the same relative position which it had in respect to the plate P on the occasion of the impression.

Making the luminous rays describe a path exactly opposite to that which they had in producing the picture, it is evident that the rays which have traversed the plate, proceed within the cone $O\,N\,N'$ (the others are cut off by the diaphragm G at the optical point o), and traverse the lenses of the magnifier through the same points of the lenses which they had traversed in their path through the plate; and, consequently, in this second passage are modified equally, but in the contrary sense, so that the new image at F is exactly equal to the focal image of the object-glass, the effects of distortion thus disappearing. Placing a micrometer at C, we can make the measures as in the direct observation of the Sun.

It is obvious that it is quite a different thing to measure the image of the Sun thus diminished by the magnifier, or to measure a photographic image produced by the object-glass; in this second case all the imperfections are augmented; one magnifies the granulation of the collodion, or of the silver, &c.; which does not happen in the first case.

These are the principal points of the method we had decided upon, and which unfortunately, will not be realised; it is just this which I regret, not to be able to realise a plan which in all probability would have succeeded.

If you approve of any of these points, for instance the method of measuring the photograms, it might be possible to employ them at some of the English stations.

Lisbon,
1874, *April* 26.

Signor Capello, in a subsequent letter, enquires whether the *personnel* of the different English expeditions is quite filled up; and, in the contrary case, whether it would be possible for him to join in any one of them; by preference in the Northern hemisphere, so as not to take him too long a time from the Observatory. He has practised magnetic observations for sixteen years.

On the Lunar Semi-diameter derived from Occultations of Stars. By E. Neison, Esq.

It has been shown from a reduction of the Greenwich observation for the years 1861–1870 that the semi-diameter given by occultations of stars at the dark limb was considerably less than Hansen's value. As a variation in the apparent semi-diameter arising from irradiation was found with instruments of different aperture, it became necessary to determine if a similar variation existed in

the correction from occultations. For this purpose the observations made at the Radcliffe Observatory with an aperture of $7\frac{1}{2}$ inches during the period 1862–1872, and those made at Cambridge with a mean aperture of 11 inches during the period 1861–1869, were compared with the Greenwich observations for 1861–1871, made with an average aperture of about 6 inches.

Seeing the great difficulty in making the observations, the small number made, and the one-sided nature of the errors, the determinations from the re-appearances are omitted.

	Disappearances at Dark Limb.	No. of Obs.	Disappearances at Bright Limb.	No. of Obs.
	"		"	
Greenwich	−1·73	42	+1·06	12
Radcliffe	−1·81	30	+0·52	13
Cambridge	−1·95	14	+0·01	3

The correction for the first class is nearly constant; while the gradually decreasing correction with increase of aperture in the second class, is exactly what is to be anticipated. It is evident that only in the disappearances at the dark limb is the full effect of the correction apparent; for at the disappearances at the bright limb, owing to the great comparative faintness of the star, there is a considerable tendency to lose sight of it before actual occultation. The very great difficulty in observing the reappearances at the exact instant of becoming visible, and not a few seconds later, tends, like the one-sided nature of the resulting error, to increase the real occultation semi-diameter, and so lessen very much their value.

Combining now the whole series of 176 observations, we have, as the correction to Hansen's semi-diameter:

	Correction.	No. of Obs.
	"	
Disappearances at the Dark Limb	−1·80	86
„ „ Bright Limb	+0·70	28
Reappearances at the Dark Limb	+0·13	29
„ „ Bright Limb	+1·84	33

Owing to the irradiation at the lunar limb, the actual mean diameter is not known; but the minimum value it can have, is found from observations of Solar eclipses where the effects of irradiation are reversed.

As the effect of irradiation appears to vary directly as the difference in brilliancy of the two objects, it is to be expected that its effects would be greater in the case of the dark Moon on the Sun's disk than in that of the bright Moon against the sky. Assuming that of the difference observed between the diameters of the bright and of the dark eclipsed Moon a little over one-half is due to solar irradiation, as the contact is then greatest, the minimum

lunar semi-diameter can be found. For the observations of the Eclipses of July 18, 1860, and December 21, 1870, at Greenwich with the great Equatoreal, give as the minimum value for the mean lunar semi-diameter, Hansen's value less 0″·13 and less 0″·20 respectively. This value being still over 1″·60 greater than the occultation semi-diameter, it would appear that irradiation fails to account for the difference found to exist.

To compare these results with those obtained by the Astronomer Royal (*Greenwich Observations*, 1864), they were reduced to the same parallax and semi-diameter there used. As the earlier observations of the series employed by the Astronomer Royal are by no means so good as those made at later periods, when the Altazimuth has been in use, they are omitted. Employing then those made during the period 1850–1860, with an average aperture of a little over four inches, and comparing with the results of the later Greenwich, and the Radcliffe and Cambridge observations, we have, for the disappearances at the dark limb:

Greenwich, 1850–1860	= −2″·45	Greenwich, 1861–1871	= −2″·87
Radcliffe, 1862–1872	= −2″·81	Cambridge, 1861–1869	= −2″·87

Finally combining the whole four series, we have, as reliable values, from 303 observations, reduced from good lunar places, and with fine instruments, for the correction to the telescopic semi-diameter employed by the Astronomer Royal:

Disappearances at the Dark Limb	= −2″·74	from	143 obs.
„ „ Bright Limb	= −1″·29	„	46 obs.
Reappearances at the Dark Limb	= −1″·08	„	58 obs.
„ „ Bright Limb	= +0″·46	„	56 obs.

London,
1874, 30*th March.*

The Solar and Planetary Systems. By Maxwell Hall, Esq.

The author states Bode's law as follows:—"In the solar and planetary systems the mean distances of the planets and satellites do not greatly differ in value from the terms of the series:

$$4\lambda,\ 7\lambda,\ 10\lambda,\ 16\lambda,\ 28\lambda,\ 52\lambda,\ 100\lambda,\ 196\lambda,\ 388\lambda,\ \&c.,$$

where λ has different values in different systems. But there may be more than one, or there may be no planet or satellite near any of the above theoretical distances." And he then proceeds to determine λ in miles for the planetary system, and for the Jovian, Saturnian, and Uranian satellite systems respectively.

Some of the numerical coincidences are very close, thus in the Uranian system, taking the distances to be 7λ, 10λ, 16λ, and 28λ, the first three satellites give $\lambda = 17600$, 17100, and 17600 miles respectively, (but the fourth satellite gives $\lambda = 13400$ miles).

He then states a second proposition: "Twice the unit of length in any system is approximately equal to that distance which corresponds to the period of rotation of the central body of that system," or say

$$\lambda = 1580\, M^{\frac{1}{3}} P^{\frac{2}{3}}$$

where M = mass of central body in terms of the mass of the earth, P period of the axial rotation in hours, λ in miles as before. It thus appears that dividing the value of λ for any system by the value of $M^{\frac{1}{3}} P^{\frac{2}{3}}$ for the central body of that system, the quotient should be 1580. For the Solar, Jovian, and Saturnian systems, the quotients are 1790, 1340, 1720, mean 1620. For the Earth $\lambda = 13100$ miles; so that regarding the Moon as a fourth satellite (the three interior ones missing) its theoretical distance is 210,000 miles. The paper concludes with some considerations as to M. Lescarbault's planet *Vulcan*.

Memorandum of Observations of Jupiter made during the month of April 1874. By John Brett, Esq.

I wish to call the attention of the Fellows to a particular feature of *Jupiter's* disk, which appears to me very well defined at the present time, and which seems to afford evidence respecting the physical condition of the planet.

The large white patches which occur on and about the equatoreal zone, and interrupt the continuity of the dark belts, are well known to all observers, and the particular point in connection with them, to which I beg leave to call attention is, that *they cast shadows;* that is to say, the light patches are bounded on the side farthest from the Sun by a dark border shaded off softly towards the light, and showing in a distinct manner that the patches are projected or relieved from the body of the planet.

The evidence which this observation is calculated to afford refers to the question whether the opaque body of the planet is seen in the dark belts or the bright ones, and points to the conclusion that it is not seen at all in either of them, but that all we see of *Jupiter* consists of semi-transparent materials.

The particular fact from which this inference would be drawn is, that the dark sides of the suspended or projected masses are not sufficiently hard or sharply defined for shadows falling upon an opaque surface, neither are they sharper upon the light background than upon the dark.

The laws of light and shade upon opaque bodies are very simple and very absolute; and one of the most rudimentary of them is, that every body has its light, its shade, and its shadow, the relations between which are constant; and that the most conspicuous and persistent edge or limit in this association of elements is the boundary of the shadow; the shadow being radically different from the shade in that its intensity is uniform throughout in any given instance, and is not affected by the form of the surface on which it is cast, whereas the shade is distinguished by attributes of an opposite character.

Now if the dark spaces adjoining the light patches on *Jupiter*, which I have called shadows, are not shadows at all, but shades, it is obvious that the opaque surface of the planet on which the shadows should fall is concealed; whereas if they are shadows, their boundaries are so soft and undefined, as to lead to the conclusion that they are cast upon a semi-transparent body, which allows the shadow to be seen indeed, but with diminishing distinctness towards its edge, according to the acuteness of its angle of incidence.

Either explanation of the phenomenon may be the true one, but they both lead to the same conclusion, viz., that neither the dark belts nor the bright ones are opaque, and that if *Jupiter* has any nucleus at all, it is not visible to us.

It is obvious that the phenomenon I have described would not be visible at the time of the planet's opposition, and the first occasion on which I noticed it was the night of the 16th of April last. The drawing* which accompanies this memorandum represents that particular observation; but since that date I have seen it even more distinctly on several occasions, and I venture to remark, that the time of opposition may prove to be as unfavourable for examining *Jupiter* as it is for the Moon.

The instrument used was a $9\frac{1}{4}$-inch silvered reflector, with achromatic eye-pieces; the power usually found most effective being 400.

By the kind invitation of Mr. Lassell, I had an opportunity on the 20th of April of examining the disk with his 20-foot reflector of 24 inches aperture, and I found this large instrument confirm my impressions concerning the shadows in the most satisfactory manner.

38 *Harley Street, W.*

Bright Spots on Jupiter. By Joseph Gledhill, Esq., F.G.S.

These curious and beautiful objects were finely seen here about midnight, April 23rd. They lay just within the shading which surrounded the south pole of *Jupiter*. Only three were seen. They seemed quite round, about the size of Sat. I., when fairly

* This drawing was exhibited at the Meeting.—Ed.

within the disk, were sharply defined, and quite as bright as any of the bright parts of the surface of the planet.

To the east of them there was a "break" or "gap" in the northern edge of the shading about the pole; and, on looking over the sketches and notes made here since 1869, I find that these spots have always, perhaps, been accompanied by a "gap" in the dusky band in which they lay. Sometimes the spots preceded the "gap" in position; sometimes they followed it; and occasionally some lay on both sides of it.

This double phenomenon was observed here in November and December 1869, in January, October, and November 1870, in January 1872; and now it appears again.

I believe that such bright spots have but rarely been seen to the north of the equator of *Jupiter*.

Mr. Edward Crossley's Observatory,
Skircoat, Halifax,
1874, *April* 24.

Discovery of Minor Planet (137).

This planet was discovered by M. Palisa, at Pola, on April 21st, its position being—

	Mean Time at Pola.	R.A.	N.P.D.
	h m	h m s	° ′
1874 April 21	11 56	13 20 16	98 17

An observation made at Berlin on the following day gave—

	Mean Time at Berlin.	R.A.	N.P.D.
	h m	h m s	° ′
April 22	10 7	13 19 38	98 9·4

The planet was of the 12th magnitude.

Discovery and Elements of Comet II. 1874, *and Comet III.* 1874.

The first of these Comets was discovered by Dr. Winnecke, at Strasburg, on April 11th, its position being—

	Mean Time at Strasburg.	Apparent R.A.	Apparent N.P.D.
	h m	h m s	° ′
1874 April 11	15 30	21 23 8	96 56

From observations made at Kremsmünster and Vienna on

April 12th, at Leipzig on April 17th, and at Vienna on April 20th, the following elements have been calculated by Professor E. Weiss :—

T = 1874 March 13·99342 Berlin Mean Time.

π =	245° 53′ 14″	Mean Equinox 1874·0
Ω =	274 6 44	
i =	148 24 42	

log q = 9·94743

A comparison of the middle observation with the corresponding place calculated from these elements gives—

$$\Delta\, \alpha \cos\delta = -0^{s}\cdot75$$
$$\Delta\, \delta = -2''\cdot5$$

The third Comet of 1874 was first seen by M. Coggia, at Marseilles, on April 17th, its observed place on that day being—

		Mean Time at Marseilles.	Apparent R.A.	Apparent N.P.D.
		h m s	h m s	° ′ ″
1874	April 17	9 29 43·9	6 28 7·47	20 2 38·3
	17	13 8 42·0	6 27 57·80	20 3 16·6

From the observations made at Marseilles on April 17th, and two Twickenham observations of April 28 and May 9, Mr. Plummer has calculated the following elements :—

T	July 6·38065 G.M.T.	
	° ′ ″	
π	270 20 36·4	App. Eq. May 0
Ω	117 34 13·8	
i	64 29 28·2	
log q	9·8234364	

Motion direct.

The middle observation is thus represented :

C − O, long. + 7′·6
lat. 9·0

The elements show a similarity with those of Comet II. 1737.